

Analytic theory of finite asymptotic expansions in the real domain.

Part I: two-term expansions of differentiable functions.

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Abstract. It is our aim to establish a general analytic theory of asymptotic expansions of type

$$(*) \quad f(x) = a_1\phi_1(x) + \cdots + a_n\phi_n(x) + o(\phi_n(x)), \quad x \rightarrow x_0,$$

where the given ordered n -tuple of real-valued functions (ϕ_1, \dots, ϕ_n) forms an asymptotic scale at $x_0 \in \overline{\mathbb{R}}$. By analytic theory, as opposed to the set of algebraic rules for manipulating finite asymptotic expansions, we mean sufficient and/or necessary conditions of general practical usefulness in order that $(*)$ hold true. Our theory is concerned with functions which are differentiable $(n-1)$ or n times and the presented conditions involve integro-differential operators acting on f, ϕ_1, \dots, ϕ_n . We essentially use two approaches; one of them is based on canonical factorizations of n th-order disconjugate differential operators and gives conditions expressed as convergence of certain improper integrals, very useful for applications. The other approach, valid for $(n-1)$ -time differentiable functions starts from simple geometric considerations (as old as Newton's concept of limit tangent) and gives conditions expressed as the existence of finite limits, as $x \rightarrow x_0$, of certain Wronskian determinants constructed with f, ϕ_1, \dots, ϕ_n . There is a link between the two approaches and it turns out that the integral conditions found via the factorizational approach have striking geometric meanings. Our theory extends to general expansions the theory of polynomial asymptotic expansions thoroughly investigated in a previous paper. In the first part of our work we study the case of two comparison functions ϕ_1, ϕ_2 . The theoretical background for the two-term theory is much simpler than that for $n \geq 3$ and, in addition, it is unavoidable to separate the treatments as the two-term formulas must be explicitly written lest they become unreadable.

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1. Introduction

Aim of our work is the establishing of a fairly complete theory of asymptotic expansions of type

$$(1.1) \quad f(x) = a_1\phi_1(x) + \cdots + a_n\phi_n(x) + o(\phi_n(x)), \quad x \rightarrow x_0, \quad n \geq 2,$$

where the comparison functions ϕ_i are supposed known in a neighborhood of x_0 and forming an asymptotic scale at x_0 , i.e.

$$(1.2) \quad \phi_1(x) \gg \phi_2(x) \gg \cdots \gg \phi_n(x), \quad x \rightarrow x_0.$$

We deal with real-valued functions of one real variable. The simplest and first historical example of a relation (1.1)-(1.2), as all of us know, is Taylor's formula for which we have at disposal not only the elementary rules for manipulating the corresponding asymptotic relations but also the various forms of Taylor's theorem which give simple analytic conditions on f sufficient for the validity of Taylor's formula of a certain order either with a simple asymptotic estimate or with some representation of the remainder. As far as general asymptotic expansions are concerned the current state of affairs is that we are able to perform practical manipulations, see e.g. Dieudonné [2; ch. III], but no analogue of Taylor's formula is explicitly stated in the literature except in the case of expansions in real powers, i.e. $\phi_i(x) = x^{\alpha_i}$. In [5] the author collected and systematized various scattered results concerning polynomial expansions

$$(1.3) \quad f(x) = a_n x^n + \cdots + a_1 x + a_0 + o(1), \quad x \rightarrow +\infty,$$

with an eye to highlight the geometric approach and to link different approaches by a unique thread. In [6; 7] the author developed a theory for expansions in real powers

$$(1.4) \quad f(x) = a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n} + o(x^{\alpha_n}), \quad x \rightarrow +\infty; \quad \alpha_1 > \cdots > \alpha_n,$$

with the aim of obtaining complete and applicable results about the formal differentiation of (1.4), results not obtainable by any of the classical approaches used for formal differentiation of the asymptotic relations $f(x) = O(x^\gamma)$ or $f(x) = o(x^\gamma)$.

Now we intend to develop a complete theory of expansions (1.1)-(1.2) for differentiable functions. Our title “*Analitic theory of ...*” is meant as opposed to the “*algebraic theory*” i.e. the set of rules for manipulating finite asymptotic expansions. Our work deals with functions which are differentiable a sufficient number of times and the exhibited sufficient and/or necessary conditions involve certain differential operators in the same way as Taylor’s formula involves n th-order derivatives. Different approaches are used and complementary results are obtained but there is one guiding thread: the theory of Pólya-Mammana factorizations of linear ordinary differential operators in its latest developments, due to Trench [13] and the author [3; 4], concerning canonical factorizations. The n -tuple (ϕ_1, \dots, ϕ_n) is subject to the practically mild restriction of forming a Chebyshev system on a one-sided neighborhood of x_0 and this yields “natural” generalizations of Taylor’s formula. Adapting on old method by Haupt [10] we find a geometric characterization of a certain asymptotic situation involving both (1.1) and suitable relations for the derivatives of f .

Our exposition is split in two parts, according as $n = 2$ or $n \geq 3$, for technical and practical reasons. In fact the theory for $n \geq 3$ requires the results for $n = 2$ as some proofs are by induction on n ; moreover statements and formulas for $n = 2$ must be written out explicitly to avoid misinterpretations of the more complicated higher-order formulas.

Propositions are numbered consecutively irrespective of their labelling as theorem, lemma etc..

2. Various approaches to the theory

For a general two-term expansion

$$(2.1) \quad f(x) = a_1\phi_1(x) + a_2\phi_2(x) + o(\phi_2(x)), \quad x \rightarrow x_0,$$

where ϕ_1, ϕ_2 do not vanish on a suitable deleted neighborhood of x_0 and $\phi_1(x) \gg \phi_2(x)$, $x \rightarrow x_0$, we have the characterizing relations

$$(2.2) \quad a_1 = \lim_{x \rightarrow x_0} f(x)/\phi_1(x); \quad a_2 = \lim_{x \rightarrow x_0} [f(x) - a_1/\phi_1(x)]/\phi_2(x),$$

and we know that they can seldom be used in meaningful applications other than elementary cases. For this reason we look for sufficient, and possibly necessary, conditions of a quite different nature for (2.1) to hold. We point out four different approaches which are fit for the n -term theory as well.

1. The naive approach. If we try to apply L’Hospital’s rule to evaluate the second limit (2.2), writing

$$(f - a_1\phi_1)/\phi_2 \equiv \frac{f/\phi_1 - a_1}{(\phi_2/\phi_1)},$$

we see that a sufficient condition for (2.2) to hold is the existence as finite numbers of the first limit (2.2) and of

$$(2.3) \quad a_2 := \lim_{x \rightarrow x_0} \frac{(f/\phi_1)'}{(\phi_2/\phi_1)'}$$

We label this approach as “naive” because its elementary idea leads us to replace the second limit (2.2) by a seemingly more complicated limit: maybe this approach is a blind alley and we should better try other paths. However it is elementary to prove that the existence of the limit (2.3) implies the existence of the first limit (2.2) due to condition $\phi_2(x)/\phi_1(x) = o(1)$, $x \rightarrow x_0$, and to condition “ $(\phi_2(x)/\phi_1(x))'$ strictly one-signed on a neighborhood of x_0 ”, which is necessary for the application of L'Hospital's rule. Hence the pair of conditions (2.2) is implied by the single condition (2.3) defining a_2 independently from a_1 , at least under the additional restriction on the sign of $(\phi_2/\phi_1)'$. So far our investigation is nothing but an elementary exercise but we shall point out the asymptotic, the geometric and the analytic meaning of (2.3).

2. The algebraic approach: formal differentiation of asymptotic expansions.

Let us examine the case wherein the remainder in (2.1) is identically zero i.e. the given function f coincides, at least in a neighborhood of x_0 , with a “generalized polynomial”

$$(2.4) \quad f(x) \equiv a_1\phi_1(x) + a_2\phi_2(x).$$

Besides a_1 , defined by the first relation (2.2), we may express a_2 independently from a_1 through a suitable differential operator. In fact we get in sequence from (2.4):

$$(2.5) \quad f(x)/\phi_1(x) = a_1 + a_2(\phi_2(x)/\phi_1(x)),$$

$$(2.6) \quad (f(x)/\phi_1(x))' = a_2(\phi_2(x)/\phi_1(x))',$$

$$(2.7) \quad a_2 \equiv \frac{(f(x)/\phi_1(x))'}{(\phi_2(x)/\phi_1(x))'},$$

provided the last expression takes a meaning on some interval. We now apply the same procedure to (2.1) first obtaining

$$(2.8) \quad f(x)/\phi_1(x) = a_1 + a_2(\phi_2(x)/\phi_1(x)) + o(\phi_2(x)/\phi_1(x)), \quad x \rightarrow x_0,$$

and then we conjecture that formal differentiation both sides of (2.8) may yield

$$(2.9) \quad (f(x)/\phi_1(x))' = a_2(\phi_2(x)/\phi_1(x))' + o((\phi_2(x)/\phi_1(x))'), \quad x \rightarrow x_0,$$

at least under “reasonable conditions”. Relation (2.9) is an equivalent formulation of (2.3) but our procedure leads us to interpret (2.9) as a relation obtained from (2.1) by formal application of a certain first-order differential operator. We have already mentioned that (2.3) implies (2.2); our present approach suggests other natural questions:

(i) Does the existence of the limit (2.3) characterize the pair (2.8)-(2.9), i.e. the pair (2.1)-(2.9)?

(ii) Does this contingency occur in meaningful situations or does it occur in exceptional situations only?

(iii) What about the limit

$$(2.10) \quad \lim_{x \rightarrow x_0} \frac{(f(x)/\phi_2(x))'}{(\phi_1(x)/\phi_2(x))'}$$

whimsically obtained from (2.3) by interchanging the roles of ϕ_1, ϕ_2 ? Is it an unnatural quantity to be taken into consideration or has it a meaning in our context?

All these questions will receive answers in this paper. Once again notice that the present approach considers the expansion (2.1) not by itself but matched to (2.9), which is obtained by a certain process of formal differentiation.

3. The analytic or factorizational approach. The idea is to use integro--differential representations of f from which one can easily infer sufficient and/or necessary conditions for (2.1) to hold in the very same way as an expansion of any of the simple types

$$(2.11) \quad f(x) = a_1 + a_2x + o(x), \quad x \rightarrow 0; \quad f(x) = a_1x + a_2 + o(1), \quad x \rightarrow +\infty,$$

can be studied starting either from the familiar representation

$$(2.12) \quad f(x) = c_1x + c_2 + \int_T^x dt \int_T^t f''(\tau) d\tau$$

or from the less usual representation

$$(2.13) \quad f(x) = \bar{c}_1x + \bar{c}_2 + x \int_T^x t^{-2} dt \int_T^t \tau f''(\tau) d\tau.$$

The procedure goes as follows. Consider a second-order linear ordinary differential operator L whose null-set coincides with $\text{span}(\phi_1, \phi_2)$ and which can be explicitly inverted. The most favourable circumstance is whenever L admits of a factorization

$$(2.14) \quad L[u] \equiv p_2[p_1(p_0u)]'$$

with suitable nowhere-vanishing functions p_i ; in this case one can write down a corresponding integral representation of f in terms of $L[f]$:

$$(2.15) \quad f(x) = c_1\phi_1(x) + c_2\phi_2(x) + \frac{1}{p_0(x)} \int_T^x \frac{dt_1}{p_1(t_1)} \int_T^{t_1} \frac{L[f(t_2)]}{p_2(t_2)} dt_2.$$

Now one tries to obtain analytic characterizations of (2.1), or of the pair (2.1)-(2.9), or of other pairs of asymptotic relations via integrability conditions on $L[f]$. This approach automatically gives integral representations of the remainders. An exhaustive investigation involves the two types of canonical factorizations available for L and described in §3.

4. The geometric approach. The idea is nothing but Newton's concept of "limit tangent" to the graph of a function as the point of contact goes to infinity. The straight line tangent to the graph of f at a generic point $(t, f(t))$ is represented by equation

$$(2.16) \quad y = f(t) + f'(t)(x - t) \equiv f'(t)x + [f(t) - tf'(t)],$$

and its limit position as $t \rightarrow +\infty$ may be defined as the line $y = a_1x + a_2$ where

$$(2.17) \quad a_1 := \lim_{t \rightarrow +\infty} f'(t); \quad a_2 := \lim_{t \rightarrow +\infty} [f(t) - tf'(t)].$$

A satisfying treatment goes back to the German geometer Haupt [8] almost one century ago. Applying the same idea to (2.1) one first chooses, among all linear combinations $c_1\phi_1(x) + c_2\phi_2(x)$, that special one which has a first-order contact with f at a generic point t and which is characterized by certain coefficients $c_1 = f_1^*(t)$, $c_2 = f_2^*(t)$. Then one investigates relationships between the existence, as finite numbers, of any one or both of the limits $\lim_{t \rightarrow x_0} f_i^*(t)$, $i = 1, 2$, and the validity of the expansion (2.1). Such a procedure provides analytic characterizations of various asymptotic situations by means of geometrically-meaningful quantities; the $\lim_{t \rightarrow x_0} f_2^*(t)$ turns out to be more meaningful than the $\lim_{t \rightarrow x_0} f_1^*(t)$. The elementary case $(\phi_1(x), \phi_2(x)) \equiv (1, x)$ as well as the general polynomial case $(\phi_1(x), \dots, \phi_n(x)) \equiv (1, x, \dots, x^{n-1})$, studied in [5], suggest the introduction of another meaningful geometric quantity $F^*(t)$, definition 4.1 below, which represents the ordinate of the intersection point between the graph of the osculating curve $y = f_1^*(t)\phi_1(x) + f_2^*(t)\phi_2(x)$ at the generic point $(t, f(t))$ and a fixed vertical line $x = T$. The main result states the equivalence of the existence as finite quantities of any one of the following three limits:

$$(2.18) \quad \lim_{t \rightarrow x_0} F^*(t) ; \lim_{t \rightarrow x_0} f_2^*(t) ; \lim_{t \rightarrow x_0} (f_1^*(t), f_2^*(t)).$$

This contingency defines the “limit” of the osculating curve and is also equivalent to the pair (2.1)-(2.9).

* * *

Following the third and fourth approaches we shall give substance to the first and second approaches. The quantity $(f(t)/\phi_1(t))' / (\phi_1(t)/\phi_2(t))'$, found in the naive approach, turns out to be the coefficient $f_2^*(t)$ whereas the “whimsical” quantity appearing in the limit (2.10) is the coefficient $f_1^*(t)$: see §4. Moreover, if all the involved functions are supposed to be of class $AC^1(I)$, i.e. endowed with first-order derivatives absolutely continuous on a neighborhood I of x_0 , then $(f_i^*)'(t) = q_i(t) \cdot L[f(t)]$, $i = 1, 2$, where q_i are suitable nowhere-vanishing functions and L is a second-order linear ordinary differential operator such that $\ker L = \text{span}(\phi_1, \phi_2)$. Expressing f_i^* as integral operators acting on $L[f]$ we transmute all the geometric conditions into simpler and practically useful analytic conditions. Last, but not least, a unique theoretical link is found for all the approaches, namely the theory of canonical factorizations of second-order differential operators.

The theory we shall develop for two-term expansions contains all the essential ideas for the n -term expansions. In the second part of our work we shall point out the contributions of other authors to the n th-order theory, especially Kudryavtsev’s Lagrangian approach (which is nothing but the geometric approach) to a larger class of expansions.

3. Basic assumptions and preliminary material

In studying (2.1) the real-valued functions ϕ_1, ϕ_2, f are supposed to be defined on a deleted one-sided neighborhood I of x_0 and, for definiteness, we suppose that I is left-sided and $x_0 \leq +\infty$. Every limit process “ $\lim_{x \rightarrow x_0} \dots$ ” stands for “ $\lim_{x \rightarrow x_0, x \in I} \dots$ ”. If I is any interval in \mathbb{R} the symbol $AC^k(I)$ denotes the space of all f ’s such that $f \in C^k(I)$ and $f^{(k)}$ is absolutely continuous on every compact interval of I : i.e. $f \in AC^k(I) \Leftrightarrow f^{(k)} \in AC(I)$. $AC^0(I) \equiv AC(I)$. Whenever $f \in AC(I)$, writing “ $\lim_{x \rightarrow x_0} f'(x)$ ” stands for “ $\lim_{x \rightarrow x_0, x \in \tilde{I}} f'(x)$ ” where \tilde{I} is the subset of I where f' exists as a finite number; when

applying L'Hospital's rule in such a context we always use Ostrowski's version [11] valid for absolutely continuous functions. The symbols $f \in L^1(I)$, $f \in L^1_{\text{loc}}(I)$ respectively denote that f is Lebesgue-summable on I or on every compact interval of I ; f integrable on $[T, x_0[$ means that $f \in L^1_{\text{loc}}[T, x_0[$ and the improper integral

$$\int_T^{x_0} f \equiv \int_T^{\rightarrow x_0} f := \lim_{x \rightarrow x_0} \int_T^x f \quad \text{converges.}$$

$\overline{\mathbb{R}}$ is the extended real line, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Basic assumptions on (ϕ_1, ϕ_2) :

$$(3.1)_1 \quad \phi_1, \phi_2 \in C^1(I) ; I := [T, x_0[, \quad T \in \mathbb{R} ;$$

$$(3.1)_2 \quad \phi_1(x) \gg \phi_2(x), x \rightarrow x_0 \quad \text{i.e.} \quad \phi_2(x) = o(\phi_1(x)), x \rightarrow x_0 ;$$

$$(3.1)_3 \quad \phi_1(x), \phi_2(x) \neq 0 \quad \forall x \in I ;$$

$$(3.1)_4 \quad W(x) := W(\phi_1(x), \phi_2(x)) \equiv W(\phi_1, \phi_2; x) \neq 0 \quad \forall x \in I ,$$

where $W(\phi_1, \phi_2)$ is the Wronskian determinant of ϕ_1, ϕ_2 .

Immediate consequences of assumptions (3.1) are that ϕ_1, ϕ_2 are linearly independent on I ; that ϕ_1, ϕ_2, W are strictly one-signed on I and that

$$(3.2) \quad \int_T^{x_0} (\phi_2(t)/\phi_1(t))' dt \quad \text{converges,}$$

$$(3.3) \quad \phi_2(x) = -\phi_1(x) \cdot \int_x^{x_0} (\phi_2(t)/\phi_1(t))' dt, \quad x \in I.$$

It is also known that any linear combination of ϕ_1, ϕ_2 is either $\equiv 0$ on I or has at most one zero on I : see, e.g., Coppel [1; prop. 5, p. 89].

It will be specified in §6 that, by changing the signs of ϕ_1, ϕ_2 if necessary, the three conditions (3.1)₁, (3.1)₃, (3.1)₄, state that the ordered pair of functions (ϕ_1, ϕ_2) is a Chebyshev system on I and this is the theoretical framework of our theory making possible the geometric and the analytic approaches and not only the mechanical application of L'Hospital's rule to evaluate the second limit (2.2).

Strengthened basic assumptions on (ϕ_1, ϕ_2) :

All assumptions (3.1) plus the stronger regularity condition

$$(3.4) \quad \phi_1, \phi_2 \in AC^1(I)$$

In this case, besides the above-mentioned consequences, there exists a unique second-order linear ordinary differential operator L

$$(3.5) \quad L[u] := u'' + a_1(x)u' + a_2(x)u ; a_i \in L^1_{\text{loc}}(I)$$

such that

$$(3.6) \quad \ker L = \text{span} (\phi_1, \phi_2).$$

If $\phi_1, \phi_2 \in C^2(I)$ then $a_1, a_2 \in C^\circ(I)$. Condition (3.1)₃ now implies that the operator L is disconjugate on I , Coppel [1; th. 1, p. 5], hence it admits of a Pólya-Mammana factorization on I , i.e.

$$(3.7) \quad L[u] \equiv p_2[p_1(p_0 u)']' \quad \forall u \in AC^1(I),$$

where p_0, p_1, p_2 are suitable functions strictly positive on I and satisfying the regularity conditions:

$$(3.8) \quad p_0 \in AC^1(I) ; \quad p_1, p_2 \in AC^0(I).$$

For a given factorization (3.7) any function $f \in AC^1(I)$ admits of an integral representation of type

$$(3.9) \quad f(x) = c_1 \phi_1(x) + c_2 \phi_2(x) + \frac{1}{p_0(x)} \int_{T_0}^x \frac{dt_1}{p_1(t_1)} \int_{T_0}^{t_1} \frac{L[f(t_2)]}{p_2(t_2)} dt_2, \quad x \in I,$$

where T_0 is arbitrarily chosen in I and c_1, c_2 are suitable constants depending on f, ϕ_i, p_i, T_0 .

Factorizations of the operator L . Following the terminology used in [3], factorization (3.7) is called a “canonical factorization of type (I) or of type (II), at x_0 ” according as the improper integral $\int^{x_0} (1/p_1)$ respectively diverges or converges.

Lemma 3.1 (see [4; th. 6.1]. For each solution ϕ to $L[u] = 0$ strictly positive on a left-sided neighborhood of x_0 , $J \subset [T, x_0[$, there exists an “essentially unique” factorization (3.7)-(3.8) on J such that $p_0 = 1/\phi$, namely

$$(3.10) \quad L[u] \equiv \frac{W(x)}{\phi(x)} \left[\frac{(\phi(x))^2}{W(x)} \left(\frac{u}{\phi(x)} \right)' \right]' \quad \forall u \in AC^1(J).$$

(Here “essentially unique” means that the functions p_i are determined up to constant factors).

Lemma 3.2 Factorization (3.10) is a canonical factorization of type (I) at x_0 iff

$$(3.11) \quad \phi(x) \sim c \phi_2(x), \quad x \rightarrow x_0 \quad (c \neq 0);$$

and it is a canonical factorization of type (II) at x_0 iff

$$(3.12) \quad \phi(x) \sim c \phi_1(x), \quad x \rightarrow x_0 \quad (c \neq 0),$$

where c denotes a suitable constant. Contingency (3.11) occurs iff $\phi(x) \equiv c \phi_2(x)$ on I ; hence on a fixed left-sided neighborhood of x_0 there exists only “one” canonical factorization of type (I) at x_0 .

Proof. By (3.1)₂ the two circumstances (3.11) and (3.12) are the only ones that can occur as far as the asymptotic behavior of $\phi(x)$ at x_0 is concerned. If $\phi \sim c\phi_1$ then

$$(3.13) \quad \frac{W(x)}{(\phi(x))^2} \sim c^{-2} \frac{W(\phi_1(x), \phi_2(x))}{(\phi_1(x))^2} \equiv c^{-2} (\phi_2(x)/\phi_1(x))', \quad x \rightarrow x_0,$$

and this implies, by (3.2), the convergence of the improper integral

$$(3.14) \quad \int^{x_0} W(t)(\phi(t))^{-2} dt.$$

If $\phi \sim c\phi_2$ then (3.1)₂ implies that $\phi(x) = c\phi_2(x)$ on J and relation (3.13) is replaced by

$$(3.15) \quad \frac{W(x)}{(\phi(x))^2} = -c^{-2} (\phi_1(x)/\phi_2(x))', \quad x \rightarrow x_0,$$

which implies the divergence of the integral (3.14) as $\lim_{x \rightarrow x_0} \phi_1(x)/\phi_2(x) = \pm\infty$. \square

We bring to the reader's attention the fact that for a generic operator L disconjugate on an open interval $]T, x_0[$ it may happen that there exist no solution ϕ to $L[u] = 0$ strictly positive on the whole interval $]T, x_0[$ and satisfying (3.12): just think of the operator u'' on $(-\infty, +\infty)$. However this is granted if we restrict $]T, x_0[$ to $]T + \epsilon, x_0[$ or $]T, x_0 - \epsilon[$, with an arbitrary $\epsilon > 0$, as implied by [3; th.2.2, p. 162]. Our assumption (3.1)₃ only means that we are restricting the original interval if necessary. We shall develop our theory taking advantage of both types of canonical factorizations. It is immaterial whether in a factorization, either in this paper or in practical applications, some coefficient p_i happens to be strictly negative.

* * *

The following trivial formulas are reported for the sole purpose of later references. Any of the following three notations will be used to denote the Wronskian of two functions

$$(3.16) \quad W(f, g) \equiv W(f, g; x) \equiv W(f(x), g(x)) := f(x)g'(x) - f'(x)g(x);$$

$$(3.17) \quad (f/g)' = -(g/f)' \cdot (f/g)^2 \quad \text{if } f, g \neq 0;$$

$$(3.18) \quad W(f, g) = f^2 \cdot (g/f)' \quad \text{if } f \neq 0; \quad W(f, g) = -g^2 \cdot (f/g)' \quad \text{if } g \neq 0;$$

$$(3.19) \quad \frac{d}{dx} W(f, g; x) = f(x)g''(x) - f''(x)g(x);$$

$$(3.20) \quad \begin{aligned} W(a_1 f_1 + a_2 f_2, b_1 g_1 + b_2 g_2) &= a_1 b_1 W(f_1, g_1) + a_1 b_2 W(f_1, g_2) + \\ &+ a_2 b_1 W(f_2, g_1) + a_2 b_2 W(f_2, g_2), \quad (a_i, b_i = \text{constants}). \end{aligned}$$

If f_1, f_2, g_1, g_2 are linked by the relations

$$(3.21) \quad \begin{cases} f_1(x) = a_{11}g_1(x) + a_{12}g_2(x) \\ f_2(x) = a_{21}g_1(x) + a_{22}g_2(x) \\ (a_{ij} = \text{constants}) \end{cases}$$

then

$$(3.22) \quad W(f_1(x), f_2(x)) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot W(g_1(x), g_2(x)).$$

According to (3.1)₄ the symbol $W(x)$ always refer to the Wronskian of the comparison functions ϕ_1, ϕ_2 fixed in any particular context.

4. The geometric approach

As usual we say that two functions f, g (as well as their gaphs) have a *first-order contact* at a point t_0 if $f(t_0) = g(t_0)$ and $f'(t_0) = g'(t_0)$ provided that f, g are defined on a neighborhood of t_0 and the involved derivatives exist as finite numbers. The following elementary fact will provide a basis for our discussion.

Lemma 4.1 *Let ϕ_1, ϕ_2 be two functions such that*

$$(4.1) \quad \phi_1, \phi_2 \text{ differentiable on an interval } I,$$

$$(4.2) \quad W(\phi_1(x), \phi_2(x)) \neq 0 \quad \forall x \in I.$$

In particular ϕ_1, ϕ_2 may satisfy the basic assumptions (3.1) on the interval $[T, x_0[$. If f is differentiable on I then for each $t_0 \in I$ there exists a unique function in the family $\mathcal{F} := \text{span}(\phi_1, \phi_2)$ having a first-order contact with f at t_0 . Denoting this function by $F^(x; t_0)$ we have*

$$(4.3) \quad F^*(x; t_0) = f_1^*(t_0)\phi_1(x) + f_2^*(t_0)\phi_2(x), \quad x \in I,$$

where

$$(4.4) \quad \begin{cases} f_1^*(t_0) := W(f, \phi_2; t_0)/W(t_0) \equiv (f(t)/\phi_2(t))' / (\phi_1(t)/\phi_2(t))' \big|_{t=t_0}, \\ f_2^*(t_0) := -W(f, \phi_1; t_0)/W(t_0) \equiv (f(t)/\phi_1(t))' / (\phi_2(t)/\phi_1(t))' \big|_{t=t_0}. \end{cases}$$

If $f \in \mathcal{F}$ then $F^(x; t_0) \equiv f(x)$ for any chosen t_0 .*

Definition 4.1 *In the quantity $F^*(x; t_0)$ we fix $x \in I$, say $x = T$, and consider the function*

$$(4.5) \quad F^*(t) := F^*(T; t) \equiv \phi_1(T)f_1^*(t) + \phi_2(T)f_2^*(t), \quad t \in I,$$

which we call the contact indicatrix of order one of the function f at the point t with respect to the family \mathcal{F} and the straight line $x = T$.

In the sequel we always suppose $I = [T, x_0[$; the choice $x = T$ is merely a matter of convenience; any vertical line intersecting the x -interval I can do the same. $F^*(t)$ represents the ordinate of the point of intersection between the vertical line $x = T$ and the curve $y = f_1^*(t)\phi_1(x) + f_2^*(t)\phi_2(x)$ where t is thought of as fixed. By (4.2) ϕ_1 and ϕ_2 do not vanish simultaneously hence F^* is a nontrivial linear combination of f_1^*, f_2^* . It may happen that, for some choices of T , F^* coincides with f_1^* or f_2^* , a constant factor apart, according as $\phi_2(T) = 0$ or $\phi_1(T) = 0$; this simply means that in a particular situation F^* may be a redundant quantity, otherwise F^* has its own pregnant geometric meaning. For instance if $(\phi_1, \phi_2) \equiv (x, 1)$ and $I = [0, +\infty)$ then $F^*(0; t) \equiv f_2^*(t)$. This cannot happen however if condition (3.1)₃, explicitly assumed as a matter of convenience, is satisfied.

Using (4.4) F^* may be represented as

$$(4.6) \quad F^*(x) = \frac{1}{W(x)} [\phi_1(T) \cdot W(f, \phi_2; x) - \phi_2(T) W(f, \phi_1; x)] =$$

$$= \frac{1}{W(x)} \cdot W(f(x), \phi_1(T)\phi_2(x) - \phi_2(T)\phi_1(x)) \equiv W(\Phi(x), f(x))/W(x)$$

where we have put

$$(4.7) \quad \Phi(x) := \phi_2(T)\phi_1(x) - \phi_1(T)\phi_2(x).$$

If $f \in AC([T, x_0])$, then the function F^* is defined almost everywhere on I and is Lebesgue-summable on every compact interval of I .

Lemma 4.2 (Representations of f in terms of F^*, f_1^*, f_2^*). *Under the basic assumptions (3.1), except possibly (3.1)₂, and with the foregoing notations let $f \in AC([T, x_0])$. Then*

$$(4.8) \quad \Phi(x) \neq 0 \quad \forall x \in]T, x_0];$$

$$(4.9) \quad f(x) = c\Phi(x) + \Phi(x) \cdot \int_{T_0}^x W(t)\Phi^{-2}(t)F^*(t)dt, \quad x \in]T, x_0[,$$

where T_0 is an arbitrarily fixed point in $]T, x_0[$ and c a suitable constant. Also, the following two representations are valid on the whole interval $[T, x_0[$ with suitable constants

$$(4.10) \quad f(x) = c_2\phi_2(x) - \phi_2(x) \cdot \int_T^x W(t)(\phi_2(t))^{-2}f_1^*(t)dt, \quad x \in [T, x_0[;$$

$$(4.11) \quad f(x) = c_1\phi_1(x) + \phi_1(x) \cdot \int_T^x W(t)(\phi_1(t))^{-2}f_2^*(t)dt, \quad x \in [T, x_0[.$$

Proof. Φ is a nontrivial linear combination of ϕ_1, ϕ_2 as $\phi_i(T) \neq 0$ ($i = 1, 2$); hence, as mentioned after formula (3.3), Φ has at most one zero, namely $x = T$, and (4.8) follows. On $]T, x_0[$ representation (4.6) can be written as

$$(4.12) \quad F^*(x) = \frac{\Phi^2(x)}{W(x)} \left(\frac{f(x)}{\Phi(x)} \right)' \quad a.e. \text{ on }]T, x_0],$$

from whence (4.9) follows as $W \cdot \Phi^{-2}$ is continuous and f/Φ is absolutely continuous. Representations (4.10), (4.11) are similarly obtained and are valid on $[T, x_0[$ by (3.1)₃. \square

So far the growth-order relation (3.1)₂ has played no role but it will play an essential one in obtaining our main results.

We shall characterize the contingencies wherein each of the functions $f_i^*(x)$ and $F^*(x)$ admits of a finite limit as $x \rightarrow x_0$ by means of suitable pairs of asymptotic expansions of f and f' with respect to the asymptotic scale (ϕ_1, ϕ_2) .

Hypotheses for the three theorems in this section:

$$(4.13) \quad \begin{cases} \text{(i)} & \text{the basic assumptions (3.1) about the pair of comparison} \\ & \text{functions } \phi_1, \phi_2; \\ \text{(ii)} & \text{a function } f \in AC([T, x_0]); \\ \text{(iii)} & \text{notations as in lemmas 4.1 – 4.2 and definition 4.1.} \end{cases}$$

Theorem 4.3 (The contingency: $\lim_{x \rightarrow x_0} f_1^*(x) = a_1$). (I) *The following are equivalent properties:*

(i) *There exists a finite limit*

$$(4.14) \quad \lim_{x \rightarrow x_0} f_1^*(x) \equiv a_1.$$

(ii) *It holds true the asymptotic relation*

$$(4.15) \quad \left(\frac{f(x)}{\phi_2(x)} \right)' = a_1 \left(\frac{\phi_1(x)}{\phi_2(x)} \right)' + o \left(\left(\frac{\phi_1(x)}{\phi_2(x)} \right)' \right), \quad x \rightarrow x_0.$$

(iii) *It holds true the pair of asymptotic relations*

$$(4.16) \quad \begin{cases} f(x) = a_1 \phi_1(x) + o(\phi_1(x)) \\ \left(\frac{f(x)}{\phi_2(x)} \right)' = a_1 \left(\frac{\phi_1(x)}{\phi_2(x)} \right)' + o \left(\left(\frac{\phi_1(x)}{\phi_2(x)} \right)' \right) \end{cases}, \quad x \rightarrow x_0.$$

The above equivalences are simple consequences of the identity

$$(4.17) \quad (f(x)/\phi_2(x))' = (\phi_1(x)/\phi_2(x))' f_1^*(x), \quad x \in [T, x_0[.$$

The constant a_1 in (4.14), (4.15), (4.16) is the same.

(II) *The pair of conditions*

$$(4.18) \quad \lim_{x \rightarrow x_0} f_1^*(x) = a_1; \quad \int_T^{x_0} \left(\frac{\phi_1(t)}{\phi_2(t)} \right)' [f_1^*(t) - a_1] dt \text{ convergent}$$

is equivalent to the pair of asymptotic relations

$$(4.19) \quad \begin{cases} f(x) = a_1 \phi_1(x) + a_2 \phi_2(x) + o(\phi_2(x)) \\ \left(\frac{f(x)}{\phi_2(x)} \right)' = a_1 \left(\frac{\phi_1(x)}{\phi_2(x)} \right)' + o \left(\left(\frac{\phi_1(x)}{\phi_2(x)} \right)' \right) \end{cases}, \quad x \rightarrow x_0.$$

The constant a_1 in (4.18), (4.19) is the same whereas a_2 is another suitable constant. If this is the case we have representation

$$(4.20) \quad f(x) = a_1 \phi_1(x) + a_2 \phi_2(x) - \phi_2(x) \cdot \int_x^{x_0} \left(\frac{\phi_1(t)}{\phi_2(t)} \right)' [f_1^*(t) - a_1] dt, \quad x \in [T, x_0[.$$

The following intermediary result is an essential step in proving the subsequent main theorem.

Theorem 4.4 (Characterizations of a two-term asymptotic expansion). *Under assumptions (4.13) the following are equivalent properties:*

(i) *It holds true an asymptotic expansion*

$$(4.21) \quad f(x) = a_1 \phi_1(x) + a_2 \phi_2(x) + o(\phi_2(x)), \quad x \rightarrow x_0.$$

(ii) *There exists a finite limit*

$$(4.22) \quad \lim_{x \rightarrow x_0} \frac{\phi_1(x)}{\phi_2(x)} \cdot \int_x^{x_0} W(t) (\phi_1(t))^{-2} f_2^*(t) dt \equiv \lim_{x \rightarrow x_0} \frac{\phi_1(x)}{\phi_2(x)} \cdot \int_x^{x_0} \left(\frac{\phi_2(t)}{\phi_1(t)} \right)' f_2^*(t) dt \equiv -m.$$

(iii) *There exists a finite limit*

$$(4.23) \quad \lim_{x \rightarrow x_0} \frac{\Phi(x)}{\phi_2(x)} \cdot \int_x^{x_0} W(t) (\Phi(t))^{-2} F^*(t) dt \equiv -\frac{l}{\phi_2(T)}.$$

If this is the case we have the following two representations

$$(4.24) \quad \begin{cases} f(x) = a_1 \phi_1(x) + a_2 \phi_2(x) - \phi_1(x) \cdot \int_x^{x_0} W(t) (\phi_1(t))^{-2} [f_2^*(t) - m] dt \equiv \\ \equiv a_1 \phi_1(x) + a_2 \phi_2(x) - \phi_1(x) \cdot \int_x^{x_0} \left(\frac{\phi_2(t)}{\phi_1(t)} \right)' [f_2^*(t) - m] dt, \quad x \in [T, x_0[; \end{cases}$$

$$(4.25) \quad f(x) = a_1\phi_1(x) + a_2\phi_2(x) - \Phi(x) \cdot \int_x^{x_0} W(t)(\Phi(t))^{-2}[F^*(t) - l]dt, \quad x \in]T, x_0[.$$

The validity of (4.21) may be expressed by the geometric locution: “the graph of f admits of the curve $y = a_1\phi_1(x) + a_2\phi_2(x)$ as an asymptotic curve in the family $\mathcal{F} \equiv \text{span}(\phi_1, \phi_2)$, as $x \rightarrow x_0$.” At the end of this section we suggest an expressive way of reading theorem 4.4 which for the time being looks like a technical lemma.

In general there is no immediate relationship between the numbers a_i, m and l as in the case discussed in the following theorem, one of the main results in the paper.

Theorem 4.5 (The contingency $\lim_{x \rightarrow x_0} f_2^*(x) = a_2$; characterizations of a limit tangent curve). *Let assumptions (4.13) hold true.*

(I) *The following are equivalent properties*

(i) *There exists a finite limit*

$$(4.26) \quad \lim_{x \rightarrow x_0} F^*(x) \equiv \gamma.$$

(ii) *There exists a finite limit*

$$(4.27) \quad \lim_{x \rightarrow x_0} f_2^*(x) \equiv a_2 \quad (\text{see (4.29)}_2 \text{ below}).$$

(iii) *The following two limits exist as finite numbers*

$$(4.28) \quad \lim_{x \rightarrow x_0} f_1^*(x) \equiv a_1 \quad ; \quad \lim_{x \rightarrow x_0} f_2^*(x) \equiv a_2.$$

(iv) *It holds the pair of asymptotic relations*

$$(4.29)_1 \quad f(x) = a_1\phi_1(x) + a_2\phi_2(x) + o(\phi_2(x)), \quad x \rightarrow x_0,$$

$$(4.29)_2 \quad \left(\frac{f(x)}{\phi_1(x)} \right)' = a_2 \left(\frac{\phi_2(x)}{\phi_1(x)} \right)' + o \left[\left(\frac{\phi_2(x)}{\phi_1(x)} \right)' \right], \quad x \rightarrow x_0.$$

(v) *It holds the pair of asymptotic relations*

$$(4.30)_1 \quad f(x) = a_1\phi_1(x) + a_2\phi_2(x) + o(\phi_2(x)), \quad x \rightarrow x_0,$$

$$(4.30)_2 \quad \left(\frac{f(x)}{\phi_2(x)} \right)' = a_1 \left(\frac{\phi_1(x)}{\phi_2(x)} \right)' + o \left[\frac{\phi_2(x)}{\phi_1(x)} \left(\frac{\phi_1(x)}{\phi_2(x)} \right)' \right], \quad x \rightarrow x_0.$$

(vi) *There exists a function \overline{F} , Lebesgue-summable on every compact interval of I such that*

$$(4.31)_1 \quad \overline{F}(x) = o(1), \quad x \rightarrow x_0,$$

and

$$(4.31)_2 \quad f(x) = a_1\phi_1(x) + a_2\phi_2(x) - \Phi(x) \int_x^{x_0} W(t)(\Phi(t))^{-2}\overline{F}(t)dt, \quad x \in]T, x_0[.$$

If this is in the case then

$$(4.32) \quad \overline{F}(x) = F^*(x) - \gamma \quad \text{a.e. on } I.$$

(vii) There exists a function \overline{f}_2 , Lebesgue-summable on every compact interval of I such that

$$(4.33)_1 \quad f_2(x) = o(1), \quad x \rightarrow x_0,$$

and

$$(4.33)_2 \quad f(x) = a_1\phi_1(x) + a_2\phi_2(x) - \phi_1(x) \cdot \int_x^{x_0} \left(\frac{\phi_2(t)}{\phi_1(t)} \right)' \overline{f}_2(t) dt, \quad x \in [T, x_0[.$$

If this is the case then

$$(4.34) \quad \overline{f}_2(x) = f_2^*(x) - a_2 \quad \text{a.e. on } I.$$

(II) Whenever properties in part (I) hold true then:

(viii) The family of curves whose equations with respect to cartesian coordinates x, y are

$$(4.35) \quad y = F^*(x; \xi) \equiv f_1^*(\xi)\phi_1(x) + f_2^*(\xi)\phi_2(x),$$

admits of a “limit position” as $\xi \rightarrow x_0$, namely

$$(4.36) \quad y = a_1\phi_1(x) + a_2\phi_2(x),$$

whose right-hand side is an asymptotic expansion of f , as $x \rightarrow x_0$, formally differentiable once in the sense of relation (4.29)₂. We say that the graph of f admits of a “limit tangent curve” in the family \mathcal{F} as $x \rightarrow x_0$, and this is a stronger contingency than the existence of an asymptotic curve as in theorem 4.4.

(ix) The numbers γ, a_1, a_2 are linked by relation

$$(4.37) \quad \gamma = a_1\phi_1(T) + a_2\phi_2(T)$$

whereas for the numbers m, l appearing in theorem 4.4 we have $m = a_2, l = \gamma$.

(x) They hold the two representations (4.24) and (4.25) with $m = a_2$ and $l = \gamma$.

Remarks. By (4.4) the limit relations (4.28), when written out explicitly, coincide respectively with the asymptotic relations (4.15) and (4.29)₂ which can be also written in equivalent forms as

$$(4.38) \quad W(f, \phi_2; x) = a_1W(x) + o(W(x)), \quad x \rightarrow x_0,$$

$$(4.39) \quad W(f, \phi_1; x) = -a_2W(x) + o(W(x)), \quad x \rightarrow x_0.$$

By looking at the mere formal aspect it is not self-evident that (4.29)₂ is stronger than (4.15): this follows instead from the detailed results in the foregoing theorems.

That the sole relation $(4.29)_2 \equiv (4.27)$, implies $(4.29)_1$ is a trivial consequence of (3.2) whereas relation $(4.30)_2$, which is a reinforced form of (4.15), does not generally imply $(4.30)_1$. A trivial counterexample is provided by

$$\phi_1(x) := x; \quad \phi_2(x) := 1; \quad f(x) := x + \log(\log x); \quad x_0 = +\infty.$$

* * *

Before closing this section we mention how theorem 4.4 can be given a more expressive asymptotic meaning. Let us notice that a quantity such as

$$(4.40) \quad \phi(x) \cdot \int_x^{x_0} (1/\phi(t))' f(t) dt$$

is a kind of “*weighted integral mean of f* ” and that its limit as $x \rightarrow x_0$ can be considered, the sign apart, as a “*generalized limit of $f(x)$ as $x \rightarrow x_0$* ” for the simple reason that a trivial application of L’Hospital’s rule yields

$$\lim_{x \rightarrow x_0} \frac{x \cdot \int_x^{x_0} (1/\phi(t))' f(t) dt}{(1/\phi(x))} = - \lim_{x \rightarrow x_0} f(x)$$

provided that: $\phi \in AC([T, x_0[); \phi'(x) > 0$ or $\phi'(x) < 0$ a.e.; $\lim_{x \rightarrow x_0} \phi(x) = \pm\infty$ and $\lim_{x \rightarrow x_0} f(x)$ exists in $\overline{\mathbb{R}}$.

In the very simple case $\phi(x) \equiv x$ on $[T, +\infty)$ (4.40) reduces (the sign apart) to

$$(4.41) \quad \lim_{x \rightarrow +\infty} x \cdot \int_x^{+\infty} t^{-2} f(t) dt.$$

A result by Ostowski [12; IV, pp. 65-68] states that the limit (4.41) is equivalent to the simpler limit

$$(4.42) \quad \lim_{x \rightarrow +\infty} \frac{1}{x} \cdot \int_T^x f(t) dt.$$

This limit appears here and there in the literature in problems related to the asymptotic behavior of solutions to ordinary differential equations, to the asymptotic behavior of Laplace transform and so on. In the case that f is p -periodic the limit (4.42) exists and equals the usual mean of f “ $\frac{1}{p} \int_T^{T+p} f$ ”. In general if the quantity (4.42) is defined in \mathbb{R} it may be called the “*asymptotic mean of f at $+\infty$* ”. By analogy we may label the quantity

$$(4.43) \quad - \lim_{x \rightarrow x_0} \phi(x) \cdot \int_x^{x_0} (1/\phi(t))' f(t) dt,$$

if it is defined as a real number and with the above-specified restrictions on ϕ , by the locution “*asymptotic mean of $f(x)$, as $x \rightarrow x_0$, with respect to the weight function ϕ* ”.

Of course some regularity condition on f is required to give meaning to the foregoing integrals. With this terminology the equivalence “(4.21) \Leftrightarrow (4.22)” may be reformulated as follows:

Theorem 4.4 reformulated. *Under assumptions (4.13) the function f admits of an asymptotic expansion with respect to the asymptotic scale (ϕ_1, ϕ_2) , as $x \rightarrow x_0$, iff the associated geometric quantity f_2^* admits of an asymptotic mean with respect to the weight function $\phi_1(x)/\phi_2(x)$ as $x \rightarrow x_0$.*

By further investigation it is found out that the generalized asymptotic mean (4.43) is equivalent to the standard and simpler asymptotic mean (4.42) whenever ϕ is regularly varying at $+\infty$. We shall not go into the details of this subject in this paper.

5. The factorizational approach. Estimates of the remainder

In this section our basic assumptions on ϕ_1, ϕ_2 are the strengthened ones, i.e. (3.1) plus (3.4), and $f \in AC^1(I)$. The treatment is based on canonical factorizations of the operator L defined by (3.5)-(3.6). The link between the formulas in this section and those in the preceding one is provided by the following simple fact.

Lemma 5.1. *Hypotheses: (i) the basic assumptions (3.1) and (3.4); (ii) $f \in AC^1(I)$; (iii) let $\phi(x) := c_1\phi_1(x) + c_2\phi_2(x)$ be such that $\phi(x) \neq 0$ on some interval $J \subset I$; (iv) let*

$$(5.1) \quad \tilde{f}(x) := W(f(x), \phi(x))/W(x).$$

Thesis: it holds the formula

$$(5.2) \quad \tilde{f}'(x) = -\phi(x)(W(x))^{-1} \cdot L[f(x)] \quad \text{a.e. on } J,$$

where L is the differential operator (3.5)-(3.6), whence a representation of type

$$(5.3) \quad \tilde{f}(x) = c - \int_T^x \phi(t)(W(t))^{-1} \cdot L[f(t)]dt, \quad x \in J.$$

Replacing \tilde{f} by any of the three functions f_1^*, f_2^*, F^* defined in §4 we get representations

$$(5.4) \quad f_1^*(x) = \bar{c}_1 - \int_T^x \phi_2(t)(W(t))^{-1} \cdot L[f(t)]dt, \quad x \in [T, x_0[;$$

$$(5.5) \quad f_2^*(x) = \bar{c}_2 + \int_T^x \phi_1(t)(W(t))^{-1} \cdot L[f(t)]dt, \quad x \in [T, x_0[;$$

$$(5.6) \quad F^*(x) = \bar{c} + \int_T^x \Phi(t)(W(t))^{-1} \cdot L[f(t)]dt, \quad x \in [T, x_0[.$$

Proof. Factorization (3.10) may be rewritten as

$$(5.7) \quad L[u] \equiv \frac{W(x)}{\phi(x)} \left[\frac{W(\phi(x), u)}{W(x)} \right]'$$

from whence (5.2) follows. □

The approach based on the canonical factorization of type (I) at x_0 . The “unique” factorization of type (I) at x_0 is

$$(5.8) \quad L[u] \equiv \frac{W(x)}{\phi_2(x)} \left[\frac{(\phi_2(x))^2}{W(x)} \left(\frac{u}{\phi_2(x)} \right)' \right]',$$

which gives rise to representation

$$(5.9) \quad f(x) = c_1 \phi_1(x) + c_2 \phi_2(x) + \\ + \phi_2(x) \int_T^x (\phi_2(t))^{-2} W(t) dt \int_T^t \phi_2(s) (W(s))^{-1} \cdot L[f(s)] ds, \quad x \in [T, x_0[.$$

The approach based on a canonical factorization of type (II) at x_0 . By lemma 3.2 the simplest choice of a factorization of type (II) at x_0 , in terms of the given function ϕ_1, ϕ_2 is

$$(5.10) \quad L[u] \equiv \frac{W(x)}{\phi_1(x)} \left[\frac{(\phi_1(x))^2}{W(x)} \left(\frac{u}{\phi_1(x)} \right)' \right]',$$

which gives rise to representation

$$(5.11) \quad f(x) = c_1 \phi_1(x) + c_2 \phi_2(x) + \\ + \phi_1(x) \int_T^x (\phi_1(t))^{-2} W(t) dt \int_T^t \phi_1(s) (W(s))^{-1} \cdot L[f(s)] ds, \quad x \in [T, x_0[.$$

Representations (5.4)-(5.5) give the geometric meanings of the inner integrals appearing respectively in (5.9) and (5.11); if these inner integrals are replaced by (5.4) and (5.5) we get representations practically equivalent to (4.10) and (4.11).

Representation (5.9), i.e. (4.10), is convenient to characterizing asymptotic expansions for f , matched to an asymptotic relation involving $(f/\phi_2)'$, see theorem 4.3, whereas representation (5.11), i.e. (4.11), is better fit to studying expansions for f matched to relations involving $(f/\phi_1)'$ as in theorems 4.4, 4.5. The factorizational approach yields three integral representations which allow easy characterizations of certain asymptotic expansions through integral conditions involving $L[f]$, and this is the practical usefulness of this approach.

Theorem 5.2 (Restatement of conditions appearing in theorems 4.3-4.5). *Assumptions are: (3.1) plus (3.4) and $f \in AC^1(I)$.*

(I) (Refer to theorem 4.3). Condition (4.14) for some real number a_1 is equivalent to the integral condition

$$(5.12) \quad \int_T^{x_0} \phi_2(t)(W(t))^{-1} \cdot L[f(t)]dt \quad \text{convergent},$$

from whence it follows representation

$$(5.13) \quad f_1^*(x) = a_1 + \int_x^{x_0} \phi_2(t)(W(t))^{-1} \cdot L[f(t)]dt, \quad x \in [T, x_0[.$$

The pair of conditions (4.18) holds true iff the iterated improper integral

$$(5.14) \quad \int_T^{x_0} \left(\frac{\phi_1(t)}{\phi_2(t)} \right)' dt \int_t^{x_0} \phi_2(s)(W(s))^{-1} \cdot L[f(s)]ds \quad \text{converges}.$$

(II) (Refer to theorem 4.5). Condition (4.27) for some real number a_2 is equivalent to condition

$$(5.15) \quad \int_T^{x_0} \phi_1(t)(W(t))^{-1} \cdot L[f(t)]dt \quad \text{convergent},$$

from whence we get representation

$$(5.16) \quad f_2^*(x) = a_2 - \int_x^{x_0} \phi_1(t)(W(t))^{-1} \cdot L[f(t)]dt, \quad x \in [T, x_0[.$$

Condition (4.26) for some real number γ is equivalent to

$$(5.17) \quad \int_T^{x_0} \Phi(t)(W(t))^{-1} \cdot L[f(t)]dt \quad \text{convergent},$$

which yields representation

$$(5.18) \quad F^*(x) = \gamma - \int_x^{x_0} \Phi(t)(W(t))^{-1} \cdot L[f(t)]dt, \quad x \in [T, x_0[.$$

Representations (5.13), (5.16) and (5.18) may be substituted into (4.20), (4.24) and (4.25) respectively so obtaining numerically useful formulas reported in the theorem below.

Theorem 5.3 (Representations and estimates of the remainder). *Let $f \in AC(I)$ and let its graph admit of the curve $y = a_1\phi_1(x) + a_2\phi_2(x)$ as a limit tangent curve in the family \mathcal{F} , as $x \rightarrow x_0$. Put*

$$(5.19) \quad R(x) := f(x) - a_1\phi_1(x) - a_2\phi_2(x).$$

(I) Integral representations. *In the situation of theorem (4.5) we have*

$$(5.20) \quad \begin{aligned} R(x) &= -\phi_1(x) \cdot \int_x^{x_0} W(t)(\phi_1(t))^{-2} [f_2^*(t) - a_2] dt \equiv \\ &\equiv -\phi_1(x) \cdot \int_x^{x_0} (\phi_2(t)/\phi_1(t))' [f_2^*(t) - a_2] dt, \quad x \in [T, x_0[; \end{aligned}$$

$$(5.21) \quad R(x) = -\Phi(x) \cdot \int_x^{x_0} W(t)(\Phi(t))^{-2} [F^*(t) - \gamma] dt, \quad x \in]T, x_0].$$

If ϕ_1, ϕ_2, f satisfy the stronger assumptions in theorem 5.2 then

$$(5.22) \quad R(x) = \phi_1(x) \cdot \int_x^{x_0} (\phi_2(t)/\phi_1(t))' dt \int_t^{x_0} \phi_1(s)(W(s))^{-1} \cdot L[f(s)] ds, \quad x \in [T, x_0[;$$

$$(5.23) \quad R(x) = -\phi_2(x) \cdot \int_x^{x_0} (\phi_1(t)/\phi_2(t))' dt \int_t^{x_0} \phi_2(s)(W(s))^{-1} \cdot L[f(s)] ds, \quad x \in [T, x_0[;$$

$$(5.24) \quad R(x) = \Phi(x) \cdot \int_x^{x_0} W(t)(\Phi(t))^{-2} dt \int_t^{x_0} \Phi(s)(W(s))^{-1} \cdot L[f(s)] ds, \quad x \in]T, x_0].$$

(II) Estimates. *From (5.20) and (5.21) we get respectively*

$$(5.25) \quad |R(x)| \leq |\phi_2(x)| \cdot \left(\text{ess.sup.}_{x < t < x_0} |f_2^*(t) - a_2| \right), \quad x \in [T, x_0[;$$

$$(5.26) \quad |R(x)| \leq \left| \frac{\phi_2(x)}{\phi_2(T)} \right| \cdot \left(\text{ess.sup.}_{x < t < x_0} |F^*(t) - \gamma| \right), \quad x \in [T, x_0[;$$

whereas from (5.22) and (5.24) we get respectively

$$(5.27) \quad |R(x)| \leq |\phi_2(x)| \cdot \int_x^{x_0} |\phi_1(t)(W(t))^{-1} \cdot L[f(t)]| dt, \quad x \in [T, x_0[;$$

$$(5.28) \quad |R(x)| \leq \left| \frac{\phi_2(x)}{\phi_2(T)} \right| \cdot \int_x^{x_0} |\Phi(t)(W(t))^{-1} \cdot L[f(t)]| dt, \quad x \in]T, x_0].$$

(III) Lagrange-type representations. If $f \in C^1(I)$ then for each $x \geq T$ there exists $\xi_1 \in \overline{\mathbb{R}}$, $x \leq \xi_1 \leq x_0$, such that

$$(5.29) \quad R(x) = \phi_2(x)(f_2^*(\xi_1) - a_2) \ , \quad (f_2^*(x_0) := a_2) \ ;$$

and for each $x > T$ there exists $\xi_2 \in \overline{\mathbb{R}}$, $x \leq \xi_2 \leq x_0$, such that

$$(5.30) \quad R(x) = \frac{\phi_2(x)}{\phi_2(T)}(F^*(\xi_2) - \gamma) \ , \quad (F^*(x_0) := \gamma) \ .$$

Under the stronger assumptions in theorem 5.2, (5.29) and (5.30) may be respectively written as

$$(5.31) \quad R(x) = -\phi_2(x) \cdot \int_{\xi_2}^{x_0} \phi_1(t)(W(t))^{-1} \cdot L[f(t)]dt \ , \ x \in]T, x_0[\ ;$$

$$(5.32) \quad R(x) = -\frac{\phi_2(x)}{\phi_2(T)} \cdot \int_{\xi_2}^{x_0} \Phi(t)(W(t))^{-1} \cdot L[f(t)]dt \ , \ x \in]T, x_0[\ ,$$

with the obvious agreement $\int_{x_0}^{x_0} = 0$.

Remark. Representation (5.23) comes out from (4.20) which holds true under conditions (4.18) and the weaker assumptions in theorem 4.3 granting that the curve $y = a_1\phi_1(x) + a_2\phi_2(x)$ is an asymptotic curve for the graph of f , as $x \rightarrow x_0$, but not necessarily a limit tangent curve. However numerical estimates obtained from (5.23) cannot have the simple forms reported above due to the divergence of the integral $\int^{x_0} (\phi_1/\phi_2)'$.

6. The case of generalized convex functions

The main result in this section states that: *If f is a generalized convex function with respect to the Chebyshev system (ϕ_1, ϕ_2) then the existence of an asymptotic expansion (4.21) automatically implies the existence of a limit tangent curve in the family \mathcal{F} as $x \rightarrow x_0$ (theorem 4.5) and this last contingency is even implied by the weaker relation*

$$f(x) = a_1\phi_1(x) + O(\phi_2(x)) \ , \ x \rightarrow x_0 \ .$$

Before stating the precise result we point out that the asymptotic properties discussed so far as well as those to be discussed in this section do not depend in themselves on the signs of $\phi_1, \phi_2, W(\phi_1, \phi_2)$, whereas a decisive role in the concept of convexity is played by the monotonicity of certain functions and the types of monotonicity do depend on certain signs. So it is better to make a definite agreement about the signs and this is contained in the following standard definitions of Chebyshev systems and generalized convex functions.

Definition 6.1 (Two-dimensional Chebyshev systems). *Let (ψ_1, ψ_2) be an ordered pair of continuous functions on an interval $J \in \mathbb{R}$.*

(I). (ψ_1, ψ_2) is a T -system (\equiv Chebyshev system) on J iff

$$(6.1) \quad \begin{vmatrix} \psi_1(t_1) & \psi_1(t_2) \\ \psi_2(t_1) & \psi_2(t_2) \end{vmatrix} > 0 \quad \forall t_1, t_2 \in J; \quad t_1 < t_2.$$

(II). (ψ_1, ψ_2) is a CT -system (\equiv complete Chebyshev system) on J iff, in addition to (6.1),

$$(6.2) \quad \psi_1(t) > 0 \quad \forall t \in J,$$

without any a priori restriction on the sign of ψ_2 .

(III). (ψ_1, ψ_2) is an ET -system (\equiv extended Chebyshev system) on J iff, in addition to (6.1), the following two conditions are satisfied:

$$(6.3) \quad \psi_1, \psi_2 \in C^1(J);$$

$$(6.4) \quad \begin{vmatrix} \psi_1(t) & \psi_1'(t) \\ \psi_2(t) & \psi_2'(t) \end{vmatrix} > 0 \quad \forall t \in J.$$

(IV). (ψ_1, ψ_2) is an ECT -system (\equiv extended complete Chebyshev system) on J iff all conditions (6.1), (6.2), (6.3), (6.4) are satisfied.

The above locutions are those in the book by Karlin and Studden [10; chp. I]. It is known that (ψ_1, ψ_2) is a T -system on J , except possibly for the sign of ψ_2 , iff any nontrivial linear combination of ψ_1, ψ_2 has at most one zero on J . This is stated for general T -systems and for J a compact interval in [10; th. 4.1, p.22], but a rereading of the proof shows that the argument remains unchanged for any interval. It is also known that in definition 6.1-(IV) condition (6.1) is redundant in so far as the three conditions (6.2), (6.3), (6.4) imply (6.1): see Coppel [1; prop. 5, p.89] or Karlin and Studden [10; th. 1.1, p. 376].

Definition 6.2 Let (ψ_1, ψ_2) be a Chebyshev system on an interval J ; a function $f : J \rightarrow \mathbb{R}$ is termed “convex” on J with respect to the system (ψ_1, ψ_2) iff

$$(6.5) \quad U \begin{pmatrix} \psi_1 & \psi_2 & f \\ t_1 & t_2 & t_3 \end{pmatrix} := \begin{vmatrix} \psi_1(t_1) & \psi_1(t_2) & \psi_1(t_3) \\ \psi_2(t_1) & \psi_2(t_2) & \psi_2(t_3) \\ f(t_1) & f(t_2) & f(t_3) \end{vmatrix} \geq 0$$

for each choice of $t_1, t_2, t_3 \in J : t_1 < t_2 < t_3$. It is termed “strictly convex” iff the strict sign prevails in (6.5). Whenever (6.5) is satisfied we use notation $f \in \mathcal{C}(\psi_1, \psi_2; J)$.

As a standard reference for this class of functions (with respect to an n -dimensional Chebyshev system) we again quote Karlin and Studden [10; chp. XI].

From now on in this section we shall be considering a pair of comparison functions which, besides satisfying all assumptions (3.1), form an ECT -system. According to the remarks following definition 6.1 it is enough to consider a pair (ϕ_1, ϕ_2) satisfying

$$(6.6)_1 \quad \phi_1, \phi_2 \in C^1(I) ; \quad I := [T, x_0[;$$

$$(6.6)_2 \quad \phi_1(x) \gg \phi_2(x) , \ x \rightarrow x_0^- .$$

$$(6.6)_3 \quad \phi_1(x) > 0 \ \forall x \in I ; \ \phi_2(x) \neq 0 \ \forall x \in I ;$$

$$(6.6)_4 \quad W(x) \equiv W(\phi_1(x), \phi_2(x)) > 0 \quad \forall x \in I .$$

Our main results are collected in the following two theorems: the first one dealing with monotonicity properties and the second one with asymptotic properties of two-dimensional generalized convex functions.

Theorem 6.1 (Monotonicity properties of two-dimensional generalized convex functions). *If the pair (ϕ_1, ϕ_2) satisfies all conditions (6.6) and if $f \in \mathcal{C}(\phi_1, \phi_2;]T, x_0[)$ then all the following properties hold true:*

- (i) $f \in AC]T, x_0[$.
- (ii) The three functions f_1^*, f_2^*, F^* defined by (4.4)-(4.5) are defined a.e. on $]T, x_0[$ and are monotonic on $]T, x_0[\setminus N$ where N is some Lebesgue null-set: f_2^* is increasing; f_1^* and F^* have opposite types of monotonicity and $(\text{sign} \phi_2) \cdot F^*$ is increasing.
- (iii) Any function of type $[f(x) + a_1 \phi_1(x) + a_2 \phi_2(x)]/\tilde{\phi}(x)$ is either constant or strictly monotonic on a suitable deleted neighborhood of any of the endpoints T, x_0 . Here a_k are any constants and $\tilde{\phi}$ is any nontrivial linear combination of ϕ_1, ϕ_2 .
- (iv) The following two limits exists simultaneously in $\overline{\mathbb{R}}$ and are equal

$$(6.7) \quad \lim_{x \rightarrow x_0} f(x)/\phi_1(x) = \lim_{x \rightarrow x_0} \frac{(f(x)/\phi_2(x))'}{(\phi_1(x)/\phi_2(x))'} ,$$

the roles of ϕ_1, ϕ_2 being not interchangeable: compare with theorem 4.3.

Notice that the contingency $f \in \mathcal{C}(\phi_1, \phi_2;]T, x_0[)$ can be characterized by the appropriate type of monotonicity of any of the functions f_1^*, f_2^*, F^* : see lemma 7.6 for a precise statement.

Theorem 6.2 (Asymptotic expansions of two-dimensional generalized convex functions). *Under the same assumptions on ϕ_1, ϕ_2, f as in the foregoing theorem the following facts hold true:*

- (i) We have the inference

$$(6.8) \quad f(x) = O(\phi_1(x)), \ x \rightarrow x_0 \Rightarrow \begin{cases} f(x) = a_1 \phi_1(x) + o(\phi_1(x)), \\ (f(x)/\phi_2(x))' = a_1 (\phi_1(x)/\phi_2(x))' + o(\phi_1(x)/\phi_2(x))', \end{cases}$$

for some constant a_1 : see theorem 4.3.

- (ii) To the equivalent properties (i)-(vi) listed in theorem 4.5-(I) each of the following may be added:

$$(6.9) \quad F^*(x) = O(1), \ x \rightarrow x_0 ;$$

$$(6.10) \quad f_2^*(x) = O(1), \ x \rightarrow x_0 ;$$

$$(6.11) \quad f(x) = a_1\phi_1(x) + O((\phi_2(x))), \quad x \rightarrow x_0 \quad \text{for some constant } a_1.$$

(Compare with properties (i), (ii) and (iv) in theorem 5.3).

(iii) Whenever (4.26)-(4.28) are satisfied then the following inequalities hold true

$$(6.12) \quad (\text{sign } \phi_1)(f_1^*(x) - a_1) \geq 0,$$

$$(6.13) \quad (\text{sign } \phi_2)(F^*(x) - \gamma) \leq 0,$$

$$(6.14) \quad f_2^*(x) - a_2 \leq 0,$$

for each $x \in]T, x_0[\setminus N$ where N is as in theorem 6.1, and

$$(6.15) \quad R(x) := f(x) - a_1\phi_1(x) - a_2\phi_2(x) \geq 0 \quad \forall x \in]T, x_0[.$$

Moreover if there exists a point $\xi \in]T, x_0[$ such that $R(\xi) = 0$ then $R(x) = 0 \quad \forall x \in [\xi, x_0[$.

Remarks. 1. The import of theorem 6.2 is that condition “ f/ϕ_1 bounded” implies a one-term asymptotic expansion whereas “ $(f - a_1\phi_1)/\phi_2$ bounded” implies a two-term asymptotic expansion; moreover each expansion is formally differentiable in a suitable sense (not the same in the two cases).

2. For practical applications it is important to bear in mind that under the stronger regularity conditions (3.1)-(3.4) and $f \in AC^1(I)$ we have

$$(6.16) \quad f \in \mathcal{C}(\phi_1, \phi_2; (I) \Leftrightarrow L[f(x)] \geq 0 \text{ a.e. on } I$$

where L is the operator (3.5)-(3.6): see lemma 7.6. For a function f satisfying $L[f] \geq 0$ the whole asymptotic theory developed so far admits of simpler proofs.

7. Proofs.

Lemma 7.1 (Trivial Wronskian identities). *If $\phi \in \mathcal{F} := \text{span}(\phi_1, \phi_2)$ then:*

(I) For each $i = 1, 2$

$$(7.1) \quad W(\phi, \phi_i; x) = c_i W(x) \quad , \quad x \in I,$$

where c_i is a suitable constant, possibly $c_i = 0$.

(II) If ϕ is such that $\phi(x) \neq 0 \quad \forall x \in J$, J a subinterval of I , then for at least one value of $i = 1, 2$ we have the identity

$$(7.2) \quad W(x)\phi^{-2}(x) = \bar{c}_i(\phi_i(x)/\phi(x))' \quad , \quad x \in J,$$

for a suitable constant $\bar{c}_i \neq 0$.

Proof. For $\phi = a_1\phi_1 + a_2\phi_2$ we have

$$(7.3) \quad W(a_1\phi_1 + a_2\phi_2, \phi_i) = \begin{array}{c} \nearrow_{i=1} a_2 W(\phi_2, \phi_1) = -a_2 W(\phi_1, \phi_2) \\ \searrow_{i=2} a_1 W(\phi_1, \phi_2). \end{array}$$

If $\phi \neq 0$ on J then at least one a_i is non-zero hence, by (7.3), at least one c_i in (7.1) is non-zero. For each such value of i we may write

$$(7.4) \quad W(x) = \frac{1}{c_i} W(\phi(x), \phi_i(x)) \equiv \frac{1}{c_i} \phi^2(x) (\phi_i(x)/\phi(x))', \quad x \in J.$$

□

Lemma 7.2 (Some properties of F^*).

(I) Function F^* defined by (4.4)-(4.5) remains unchanged if we add to f a function of type $c \Phi(x)$ where Φ is defined by (4.7).

(II) Condition

$$(7.5) \quad F^*(x) \equiv \gamma = \text{constant} \quad \forall x \in I$$

holds true iff $f \in \mathcal{F} := \text{span}(\phi_1, \phi_2)$.

Proof. (I) follows from (4.6). If $f = c_1\phi_1 + c_2\phi_2$ then (4.6) and (3.20) imply (7.5). Viceversa if (7.5) holds true representation (4.9) gives

$$(7.6) \quad f(x) = c\Phi(x) + \gamma\Phi(x) \cdot \int_{T_0}^x W(t)\Phi^{-2}(t)dt \stackrel{\text{by (7.2)}}{=} \bar{c}\Phi(x) + \bar{\gamma}\phi_i(x), \quad x \in I,$$

for some $i = 1, 2$ and suitable constants $\bar{c}, \bar{\gamma}$; hence $f \in \mathcal{F}$. □

Lemmas 7.1 and 7.2 do not depend on the asymptotic relation (3.1)₂; on the contrary much of the subsequent results depend on all four assumptions (3.1).

Lemma 7.3 (Some properties of Φ). Under the basic assumptions (3.1), the following are true:

$$(7.7) \quad \Phi(x) \sim \phi_2(T)\phi_1(x), \quad x \rightarrow x_0;$$

$$(7.8) \quad W(x)\Phi^{-2}(x) \sim \frac{1}{(\phi_2(T))^2} (\phi_2(x)/\phi_1(x))', \quad x \rightarrow x_0;$$

$$(7.9) \quad \int_x^{x_0} W(t)\Phi^{-2}(t)dt = -\frac{1}{\phi_2(T)} (\phi_2(x)/\Phi(x)), \quad x \in]T, x_0[;$$

$$(7.10) \quad \int_x^{x_0} W(t)\Phi^{-2}(t)dt \sim -\frac{1}{(\phi_2(T))^2} (\phi_2(x)/\phi_1(x)), \quad x \rightarrow x_0.$$

Proof. Relation (7.7) is a direct consequence of (4.7), (3.1)₂ and (3.1)₃. Relation (7.8) follows from (7.7):

$$W(x)\Phi^{-2}(x) \sim W(x)(\phi_2(T)\phi_1(x))^{-2} \equiv (\phi_2(T))^{-2} (\phi_2(x)/\phi_1(x))'.$$

Relation (11.8) implies, by (3.2), the convergence of the integral appearing in (7.9); formula (7.9) follows from

$$(7.11) \quad \begin{aligned} W(x)\Phi^{-2}(x) &\stackrel{\text{by (4.7)}}{=} \frac{1}{\phi_2(T)} W(\Phi, \phi_2; x)\Phi^{-2}(x) \equiv \\ &\equiv \frac{1}{\phi_2(T)} (\phi_2(x)/\Phi(x))', \quad x \in]T, x_0[. \end{aligned}$$

□

Proof of theorem 4.3. (I). Let (4.14) hold true. From (4.4)

$$(7.12) \quad f_1^*(x) \equiv W(f, \phi_2; x)(W(x))^{-1} = - \left(\frac{f(x)}{\phi_2(x)} \right)' (\phi_2(x))^2 (W(x))^{-1},$$

from whence (4.17) follows and from (4.17) the equivalence “(i) \Leftrightarrow (ii)” is at once inferred, together with a representation of type

$$(7.13) \quad f(x)/\phi_2(x) = c + \int_T^x (\phi_1(t)/\phi_2(t))' f_1^*(t) dt, \quad x \in I.$$

Moreover if (4.15) holds true then, by the divergence of the integral $\int^{x_0} (\phi_1/\phi_2)'$, we infer

$$\begin{aligned} f(x)/\phi_2(x) &= \bar{c} + a_1(\phi_1(x)/\phi_2(x)) + o(\phi_1(x)/\phi_2(x)) = \\ &= a_1(\phi_1(x)/\phi_2(x)) + o(\phi_1(x)/\phi_2(x)), \quad x \rightarrow x_0, \end{aligned}$$

that is to say the first relation (4.16), hence “(ii) \Leftrightarrow (iii)”.

(II). If the first condition (4.18) is satisfied we have (7.13) which we may rewrite as

$$(7.14) \quad f(x)/\phi_2(x) = \bar{c} + a_1 \frac{\phi_1(x)}{\phi_2(x)} + \int_T^x (\phi_1(t)/\phi_2(t))' [f_1^*(t) - a_1] dt.$$

If the second condition (4.18) is also satisfied then we rewrite (7.14) as

$$(7.15) \quad f(x)/\phi_2(x) = a_1 \frac{\phi_1(x)}{\phi_2(x)} + a_2 - \int_x^{x_0} (\phi_1(t)/\phi_2(t))' [f_1^*(t) - a_1] dt,$$

which implies representation (4.20) and the first relation (4.19). The second relation (4.19) holds true by part (I) of the theorem. Viceversa if (4.19) hold true then, by part (I), we have (4.14), (7.13) and (7.14). The first relation (4.19) at once implies the second condition (4.18). □

Proof of theorem 4.4. (i) \Leftrightarrow (iii): if (4.21) is true then $f = a_1\phi_1 + o(\phi_1)$; using this relation and (7.7) into representation (4.9) we get condition

$$(7.16) \quad \int^{x_0} W\Phi^{-2}F^* \quad \text{convergent},$$

and (4.9) may be rewritten as

$$(7.17) \quad f(x) = a_1\phi_1(x) + c\phi_2(x) - \Phi(x) \cdot \int_x^{x_0} W\Phi^{-2}F^* \quad x \in]T, x_0[,$$

where a_1 is the same constant as in (4.21) and c is a suitable constant. From (4.21) and (7.17) we get condition (4.23) where we have denoted the value of the limit by $-l/\phi_2(T)$ for reasons of convenience. Viceversa if (4.23) holds true we have

$$(7.18) \quad \Phi(x) \cdot \int_x^{x_0} W\Phi^{-2}F^* = -\frac{l}{\phi_2(T)}\phi_2(x) + o(\phi_2(x)),$$

and from (7.17) we get (4.21). Now, using (7.9) we rewrite (7.17) in the form

$$(7.19) \quad f(x) = a_1\phi_1(x) + \left(c + \frac{l}{\phi_2(T)}\right)\phi_2(x) - \Phi(x) \cdot \int_x^{x_0} W(t)(\Phi(t))^{-2}[F^*(t) - l], \quad x \in]T, x_0[,$$

where the last term is

$$\Phi(x) \cdot \int_x^{x_0} W\Phi^{-2}(F^* - l) = \Phi(x) \cdot \int_x^{x_0} W\Phi^{-2}F^* + \frac{l}{\phi_2(T)}\phi_2(x) \stackrel{\text{by (7.18)}}{=} o(\phi_2(x)).$$

Hence we get representation (4.25) with $a_2 = c + \frac{l}{\phi_2(T)}$.

In a similar way we show the equivalence “(i) \Leftrightarrow (ii)” and (4.24) using representation (4.11) instead of (4.9). \square

Proof of theorem 4.5. Part (I). First we notice that “(ii) \Leftrightarrow (iv)” as $(4.29)_2$ is an equivalent way of writing (4.27), by the very definition (4.4) of f_2^* , and $(4.29)_1$ is an automatic consequence of $(4.29)_2$ due to (3.2). The plan of our proof consists in proving the following inferences:

(iv) \Rightarrow (i) \Rightarrow (vi) \Rightarrow (iv); (ii) \Leftrightarrow (iii); (vi) \Rightarrow (v) \Rightarrow (i).

The equivalence “(i) \Leftrightarrow (vii)” is perfectly analogous to “(i) \Leftrightarrow (vi)”.

(iv) \Rightarrow (i). As shown at the outset of the proof of theorem 4.4 we have at our disposal representation (7.17) from which we get

$$(7.20) \quad (f(x)/\phi_1(x))' = c(\phi_2(x)/\phi_1(x))' - \left(\frac{\Phi(x)}{\phi_1(x)}\right)' \cdot \int_x^{x_0} W\Phi^{-2}F^* + \frac{W(x)}{\phi_1(x)\Phi(x)}F^*(x) =$$

$$\stackrel{\text{by (4.7)}}{=} c(\phi_2(x)/\phi_1(x))' + \phi_1(T)(\phi_2(x)/\phi_1(x))' \cdot \int_x^{x_0} W\Phi^{-2}F^* + \frac{W(x)}{\phi_1(x)\Phi(x)}F^*(x) =$$

$$\stackrel{\text{by (7.7)}}{=} c(\phi_2(x)/\phi_1(x))' + o((\phi_2(x)/\phi_1(x))') + \frac{1}{\phi_2(T)}(\phi_2(x)/\phi_1(x))'F^*(x)[1 + o(1)].$$

This trivially implies (4.26) by assumption $(4.29)_2$.

(i) \Rightarrow (vi). Relations (4.26) and (7.8) imply (7.16) and we may rewrite representation (4.9) in the form

$$\begin{aligned}
 (7.21) \quad f(x) &= \bar{c}\Phi(x) - \Phi(x) \cdot \int_x^{x_0} W\Phi^{-2}F^* \equiv \\
 &\equiv \bar{c}\Phi(x) - \gamma\Phi(x) \cdot \int_x^{x_0} W\Phi^{-2} - \Phi(x) \cdot \int_x^{x_0} W(t)\Phi^{-2}(t)[F^*(t) - \gamma]dt = \\
 &\stackrel{by(7.9)}{=} a_1\phi_1(x) + a_2\phi_2(x) - \Phi(x) \cdot \int_x^{x_0} W(t)\Phi^{-2}(t)[F^*(t) - \gamma]dt,
 \end{aligned}$$

where a_1, a_2 are suitable constants. This is (4.37) which implies the assertion in (vi).

(vi) \Rightarrow (iv). From (4.31) and (7.9) we infer that

$$(7.22) \quad \Phi(x) \cdot \int_x^{x_0} W\Phi^{-2}\bar{F} = o(\phi_2(x)), \quad x \rightarrow x_0;$$

hence representation (4.31)₂ implies (4.29)₁ with the same constants a_1, a_2 . Differentiating (4.31)₂ we get

$$(7.23) \quad (f(x)/\phi_1(x))' = a_2(\phi_2(x)/\phi_1(x))' - \left(\frac{\Phi(x)}{\phi_1(x)}\right)' \cdot \int_x^{x_0} W\Phi^{-2}\bar{F} + \frac{W(x)\bar{F}(x)}{\phi_1(x)\Phi(x)} = \dots = (4.29)_2$$

by the same calculations in (7.20) with F^* replaced by \bar{F} .

(iii) \Rightarrow (ii): obvious. (ii) \Rightarrow (iii). As noticed at the outset of the present proof relation (4.27) is nothing but relation (4.29)₂ which in turn implies (i): see the proof of “(iv) \Rightarrow (i)”. Moreover, by (4.5), both relations (4.26)-(4.27) imply the first limit (4.28).

(vi) \Rightarrow (v). We already proved that (vi) implies (4.29)₁, i.e. (4.30)₁; moreover from (4.31)₂ we get

$$(7.24) \quad (f(x)/\phi_2(x))' = a_1(\phi_1(x)/\phi_2(x))' - \left(\frac{\Phi(x)}{\phi_2(x)}\right)' \cdot \int_x^{x_0} W\Phi^{-2}\bar{F} + \frac{W(x)\bar{F}(x)}{\phi_2(x)\Phi(x)},$$

and we must estimate the last two terms on the right-hand side. We have

$$\begin{aligned}
 (7.25) \quad \left(\frac{\Phi(x)}{\phi_2(x)}\right)' \cdot \int_x^{x_0} W\Phi^{-2}\bar{F} &\stackrel{by(7.9)}{=} \left(\frac{\Phi}{\phi_2}\right)' \cdot o\left(\frac{\phi_2}{\Phi}\right) = o\left(\frac{W(\phi_2, \Phi)}{\phi_2\Phi}\right) = \\
 &\stackrel{by(7.1)}{=} o\left(\frac{W}{\phi_2\Phi}\right) \stackrel{by(7.7)}{=} o\left(\frac{W}{\phi_1\phi_2}\right) = o\left(\frac{\phi_2}{\phi_1}\left(\frac{\phi_1}{\phi_2}\right)'\right);
 \end{aligned}$$

and the last two passages show that the function $W\bar{F}(\phi_2\Phi)^{-1}$ satisfies the same asymptotic estimate as well. Substituting into (7.24) we get (4.30)₂.

(v) \Rightarrow (i). We may resort to representation (7.17) from which we get, using (4.7),

$$(7.26) \quad (f(x)/\phi_2(x))' = a_1(\phi_1(x)/\phi_2(x))' - \phi_2(T) \left(\frac{\phi_1(x)}{\phi_2(x)} \right)' \cdot \int_x^{x_0} W \Phi^{-2} F^* + \frac{W(x)F^*(x)}{\phi_2(x)\Phi(x)}.$$

For brevity we put

$$(7.27) \quad \Omega(x) := \frac{\phi_2(x)}{\phi_1(x)} \left(\frac{\phi_1(x)}{\phi_2(x)} \right)' \equiv \frac{-W(x)}{\phi_1(x)\phi_2(x)} \neq 0 \quad \forall x \in I.$$

Both (7.26) and (4.30)₂ imply

$$(7.28) \quad \frac{W(x)F^*(x)}{\phi_2(x)\Phi(x)} - \phi_2(T) \left(\frac{\phi_1(x)}{\phi_2(x)} \right)' \cdot \int_x^{x_0} W \Phi^{-2} F^* = o(\Omega(x)).$$

Now by (7.7):

$$(7.29) \quad W(x)/\phi_2(x)\Phi(x) \sim \frac{1}{\phi_2(T)} \frac{W(x)}{\phi_1(x)\phi_2(x)} \equiv -\frac{1}{\phi_2(T)} \Omega(x);$$

and, by the already-proved theorem 4.4, condition (4.23) and (7.7) imply

$$(7.30) \quad \int_x^{x_0} W \Phi^{-2} F^* = \frac{\phi_2(x)}{\phi_1(x)} \left[-\frac{l}{(\phi_2(T))^2} + o(1) \right].$$

Using (7.29) and (7.30) into (7.28) we easily get (4.26). Obviously (4.35) follows from (4.5) and representations (4.24), (4.25) hold true with $m = a_2$ and $l = \gamma$. The proof is complete. The claims in part II are contained in part I. \square

Proof of theorem 5.3. Representations (5.20) and (5.21) are nothing but (4.24) and (4.25); (5.22) and (5.24) are respectively obtained from (5.20) and (5.21) using (5.16) and (5.18); (5.23) follows from (4.20) using (5.13). In proving the estimates (5.25)-(5.26) we use in an essential way the constancy of sign of $(\phi_2/\phi_1)'$. From (5.20) we get

$$|R(x)| \leq |\phi_1(x)| \cdot \left(\text{ess.sup.}_{x < t < x_0} |f_2^*(t) - a_2| \right) \cdot \left| \int_x^{x_0} (\phi_2/\phi_1)' \right| = |\phi_2(x)| \cdot \left(\text{ess.sup.}_{x < t < x_0} |f_2^*(t) - a_2| \right);$$

whereas from (5.21), using (7.9), we get

$$\begin{aligned} |R(x)| &\leq |\Phi(x)| \left(\text{ess.sup.}_{x < t < x_0} |F^*(t) - \gamma| \right) \cdot \left| \int_x^{x_0} W(t)(\Phi(t))^{-2} dt \right| = \\ &= \left| \frac{\phi_2(x)}{\phi_2(T)} \right| \left(\text{ess.sup.}_{x < t < x_0} |F^*(t) - \gamma| \right). \end{aligned}$$

In a similar manner (5.27) and (5.28) are proved. Representation (5.29) follows from (5.20) by application of the mean-value theorem for improper integrals of type $\int_a^b \phi(t)f(t)dt$. If

$$(7.31) \quad \begin{cases} \phi \in L^1(a, b) ; \phi \geq 0 ; & f \in C^0(]a, b[) ; & a, b \in \overline{\mathbb{R}} ; \\ \text{the limits } f(a^+) \text{ and } f(b^-) \text{ exist as finite numbers ;} \end{cases}$$

then

$$(7.32) \quad \left(\int_a^b \phi \right) \cdot \left(\inf_{(a,b)} f \right) \leq \int_a^b \phi f \leq \left(\int_a^b \phi \right) \cdot \left(\sup_{(a,b)} f \right)$$

from whence

$$(7.33) \quad \int_a^b \phi f = f(\xi) \cdot \int_a^b \phi \quad \text{for a suitable } \xi \in [a, b].$$

This last equality holds true for $\phi \leq 0$ as well. In the same way (5.30) is obtained from (5.21) using (7.9). \square

For the proof of theorems 6.1, 6.2 we use some known nontrivial results about generalized convex functions. Next lemma characterizes generalized convexity on an open interval when the underlying system comprises functions of class C^1 taking account of at least one of the endpoints.

Lemma 7.4 (Karlin and Studden [10; chp XI; th. 2.1, p. 386]). *Let (ψ_1, ψ_2) be an ECT-system on the interval $[a, b[$ of the explicit form*

$$(7.34) \quad \psi_1(x) := w_1(x) \quad ; \quad \psi_2(x) := w_1(x) \cdot \int_a^x w_2(t)dt,$$

wherein

$$(7.35) \quad w_1(x), w_2(x) > 0 \quad \forall x \in [a, b[; \quad w_1 \in C^1[a, b[; \quad w_2 \in C^0[a, b[.$$

Then $f \in \mathcal{C}(\psi_1, \psi_2;]a, b[)$ iff all the following properties hold true:

- (i) f is continuous on $]a, b[$;
- (ii) f has right derivative f'_R which is right-continuous on $]a, b[$ and a left derivative f'_L which is left-continuous on $]a, b[$;
- (iii) The function

$$(7.36) \quad \rho(x) := \frac{1}{w_2(x)} \cdot D_R(f(x)/w_1(x))$$

is right-continuous and increasing on $]a, b[$, D_R denoting the right derivative.

Remark. In the quoted reference the regularity assumptions (7.35) are assumed valid on a compact interval $[a, b]$ but this is immaterial for our thesis as $f \in \mathcal{C}(\psi_1, \psi_2;]a, b[)$ iff $f \in \mathcal{C}(\psi_1, \psi_2; [\alpha, \beta])$ for each α, β : $a < \alpha < \beta < b$.

The next elementary lemma claims the invariance of the property of generalized convexity with respect to some systems to be used in the sequel.

Lemma 7.5 *Let (ϕ_1, ϕ_2) satisfy conditions (6.6) and define the following two ECT-systems on I :*

$$(7.37) \quad \bar{\psi}_1(x) := \phi_1(x); \quad \bar{\psi}_2(x) := \phi_1(x) \cdot \int_T^x (\phi_2(t)/\phi_1(t))' dt \equiv \phi_2(x) + \bar{c}\phi_1(x);$$

$$(7.38) \quad \bar{\bar{\psi}}_1(x) := |\phi_2(x)|; \quad \bar{\bar{\psi}}_2(x) := -|\phi_2(x)| \cdot \int_T^x (\phi_1(t)/\phi_2(t))' dt \equiv \\ \equiv (-\text{sign } \phi_2)\phi_1(x) + \bar{c}\phi_2(x).$$

Then $f \in \mathcal{C}(\phi_1, \phi_2; I) \Leftrightarrow f \in \mathcal{C}(\bar{\psi}_1, \bar{\psi}_2; I) \Leftrightarrow f \in \mathcal{C}(\bar{\bar{\psi}}_1, \bar{\bar{\psi}}_2; I)$.

Proof. Trivial evaluations of the determinant (6.5) for each of the three systems (ϕ_1, ϕ_2) , $(\bar{\psi}_1, \bar{\psi}_2)$, $(\bar{\bar{\psi}}_1, \bar{\bar{\psi}}_2)$ give the same value.

Lemma 7.6 (Characterizations of generalized convexity via the monotonicity of f_1^*, f_2^*, F^*). *If the pair (ϕ_1, ϕ_2) satisfies conditions (6.6) and if Φ is defined by (4.7) then $f \in \mathcal{C}(\phi_1, \phi_2;]T, x_0[)$ iff all the following properties hold true:*

- (i) f is continuous on $]T, x_0[$;
- (ii) f has a right derivative which is right-continuous and a left derivative which is left-continuous on $]T, x_0[$;
- (iii) any of the three functions

$$(7.39) \quad \rho_1(x) := \frac{D_R(f(x)/\phi_1(x))}{(\phi_2(x)/\phi_1(x))'} \equiv \frac{-W_R(f(x), \phi_1(x))}{W(x)},$$

$$(7.40) \quad \rho_2(x) := (-\text{sign } \phi_2) \frac{D_R(f(x)/\phi_2(x))}{(\phi_1(x)/\phi_2(x))'} \equiv (-\text{sign } \phi_2) \frac{W_R(f(x), \phi_2(x))}{W(x)},$$

$$(7.41) \quad \rho_3(x) := (\text{sign } \phi_2) \frac{W_R(\Phi(x), f(x))}{W(x)},$$

is right-continuous and increasing (\equiv nondecreasing) on $]T, x_0[$. Here W_R denotes the Wronskian constructed with the right derivatives.

Notice that at each point t where f is differentiable the three values $(-\text{sign } \phi_2)\rho_2(t)$, $\rho_1(t)$ and $(\text{sign } \phi_2)\rho_3(t)$ respectively coincide with $f_1^*(t)$, $f_2^*(t)$, $F^*(t)$: the three geometric quantities upon which our theory rotates.

Under the stronger regularity conditions (3.1)-(3.4) and $f \in AC^1(I)$ then $f \in \mathcal{C}(\phi_1, \phi_2;]T, x_0[)$ iff $Lf(x) \geq 0$ a.e. on $]T, x_0[$.

Proof. The statements involving either ρ_1 or ρ_2 directly follow from lemmas 7.4, 7.5. Let us prove the statement involving ρ_3 . Define

$$(7.42) \quad \tilde{\psi}_1 := |\Phi(x)| ; \tilde{\psi}_2 := |\Phi(x)| \cdot \int_{T_0}^x |(\phi_2(t)/\Phi(t))'| dt, \quad x \in]T, x_0[,$$

where T_0 is fixed in $]T, x_0[$. From (4.8), (6.6)₃ and (7.7) we infer $\text{sign } \Phi = \text{sign } \phi_2$, and from (6.6)₄ and (7.11) we infer $\text{sign } (\phi_2/\Phi)' = \text{sign } W(\Phi, \phi_2) = \text{sign } \phi_2$. Hence we rewrite (7.42) as

$$(7.43) \quad \tilde{\psi}_1 := (\text{sign } \phi_2)\Phi(x) ; \tilde{\psi}_2 := \Phi(x) \cdot \int_{T_0}^x |\phi_2(t)/\Phi(t))'| dt, \quad x \in]T_0, x[.$$

By (4.7) we explicitly get

$$(7.44)_1 \quad \tilde{\psi}_1(x) = (\text{sign } \phi_2)[\phi_2(T)\phi_1(x) - \phi_1(T)\phi_2(x)] \equiv \alpha\phi_1(x) + \beta\phi_2(x);$$

$$(7.44)_2 \quad \begin{aligned} \tilde{\psi}_2(x) &= \left[\phi_2(x) - \frac{\phi_2(T_0)}{\Phi(T_0)}\Phi(x) \right] = \\ &= \left[\frac{-\phi_2(T_0)\phi_2(T)}{\Phi(T_0)}\phi_1(x) + \left(1 + \frac{\phi_2(T_0)\phi_1(T)}{\Phi(T_0)} \right) \phi_2(x) \right] \equiv \bar{\alpha}\phi_1(x) + \bar{\beta}\phi_2(x); \end{aligned}$$

from whence

$$(7.45) \quad \begin{aligned} U \begin{pmatrix} \tilde{\psi}_1 & \tilde{\psi}_2 & f \\ t_1 & t_2 & t_3 \end{pmatrix} &= U \begin{pmatrix} \alpha\phi_1 + \beta\phi_2 & \bar{\alpha}\phi_1 + \bar{\beta}\phi_2 & f \\ t_1 & t_2 & t_3 \end{pmatrix} = \\ &= U \begin{pmatrix} \alpha\phi_1 & \bar{\beta}\phi_2 & f \\ t_1 & t_2 & t_3 \end{pmatrix} + U \begin{pmatrix} \beta\phi_2 & \bar{\alpha}\phi_1 & f \\ t_1 & t_2 & t_3 \end{pmatrix} = (\alpha\bar{\beta} + \bar{\alpha}\beta) \cdot U \begin{pmatrix} \phi_1 & \phi_2 & f \\ t_1 & t_2 & t_3 \end{pmatrix}, \end{aligned}$$

where, as trivially checked, $\alpha\bar{\beta} + \bar{\alpha}\beta = |\phi_2(T)|$. This implies that

$$f \in \mathcal{C}(\phi_1, \phi_2;]T_0, x_0[) \Leftrightarrow f \in \mathcal{C}(\tilde{\psi}_1, \tilde{\psi}_2;]T_0, x_0[).$$

Applying lemma 7.4 to the system (7.43) we get our statement, referred to the interval $]T_0, x_0[$, as the function (7.36) now becomes

$$(7.46) \quad \frac{(\text{sign } \phi_2) \cdot D_R(f(x)/\Phi(x))}{(\text{sign } \phi_2) \cdot (\phi_2(x)/\Phi(x))'} = \frac{W_R(\Phi(x), f(x))}{W(\Phi(x), \phi_2(x))} \stackrel{(4.7)}{=} \frac{W_R(\Phi(x), f(x))}{\phi_2(T)W(x)}.$$

The last statement concerning $Lf \geq 0$ directly follows from representation (5.5) and the increasing character of $f_2^* \equiv \rho_1$. \square

Proof of theorem 6.1. We report a proof of (i) as we do not have a reference for it in the literature. The monotonicity of the function (7.36) implies

$$(7.47) \quad \begin{cases} \rho & \text{continuous on }]T, x_0[\setminus \tilde{N} \text{ for a suitable countable set } \tilde{N}, \\ \rho & \text{locally bounded on }]T, x_0[. \end{cases}$$

By the continuity of w_2 we infer from (7.36) that the function $D_R(f(x)/w_1(x))$, which is defined everywhere on $]T, x_0]$ as a finite number, enjoys of the same two properties listed in (7.47) for ρ . A known result on the regularity of derivatives implies that $(f(x)/w_1(x))'$ exists as a finite number on $]T, x_0[\setminus \tilde{N}$ and is locally bounded on $]T, x_0[$. A result in real analysis, Hewitt and Stromberg [9; exercise 18.41-(d), p. 299], now implies that $f/w_1 \in AC(]T, x_0[)$ from whence $f \equiv w_1(f/w_1) \in AC(]T, x_0[)$ as $w_1 \in C^1(]T, x_0[)$.

(ii) is contained in lemma 7.6.

(iii). Put $\tilde{f} := f + a_1\phi_1 + a_1\phi_2$; then $\tilde{f} \in \mathcal{C}(\phi_1, \phi_2;]T, x_0[)$ and property (iii) of lemma 7.6 applied to \tilde{f} implies that $(\tilde{f}/\phi_i)'$, $i = 1, 2$, is either $\equiv 0$ or strictly one-signed on the specified neighborhood and the statement is proved for $\tilde{\phi} = \phi_1, \phi_2$. If $\tilde{\phi}$ is any functions in \mathcal{F} , $\tilde{\phi} \not\equiv 0$, then we know that it has at most one zero on $]T, x_0[$ hence it is $\neq 0$ on suitable deleted neighborhoods of T and of x_0 . For any such neighborhood it is obviously possible to choose another function $\tilde{\tilde{\phi}}$ in \mathcal{F} such that either the pair $(\tilde{\phi}, \tilde{\tilde{\phi}})$ or $(\tilde{\tilde{\phi}}, \tilde{\phi})$ may play the role that (ϕ_1, ϕ_2) played in lemmas 7.5 and 7.6 except possibly for some sign. Moreover either f or $-f$ belongs to the cone $\mathcal{C}(\tilde{\phi}, \tilde{\tilde{\phi}})$ in the chosen neighborhood and lemma 7.6, referred to this new context, implies that $(\tilde{f}/\tilde{\phi})'$ is either $\equiv 0$ or strictly one-signed on suitable neighborhoods of the endpoints.

(iv) follows from the monotonicity of the two quotients on the two sides of (6.7) and from L'Hospital's rule when one writes $f/\phi_1 = (f/\phi_2)/(\phi_1/\phi_2)$. The functions ϕ_1, ϕ_2 cannot be interchanged because L'Hospital's rule only works when the denominator tends to $\pm\infty$ if no information is available on the numerator. \square

Proof of theorem 6.2. (i). If $f = O(\phi_1)$ then property (iii) in theorem 6.1 implies the existence of a finite $\lim_{x \rightarrow x_0} f/\phi_1$, and the second asymptotic relation follows from (6.7).

(ii). The statement involving (6.9)-(6.10) are obvious by the monotonicity of F^* and f_2^* . Now we show that (6.11) implies (4.29)_{1,2}. Relation (4.29)₁ follows from the ultimate monotonicity of $(f - a_1\phi_1)/\phi_2$, whereas (4.29)₂ follows from the following application of L'Hospital's rule

$$(7.48) \quad a_2 = \lim_{x \rightarrow x_0} \frac{f(x) - a_1\phi_1(x)}{\phi_2(x)} \equiv \lim_{x \rightarrow x_0} \frac{\frac{f(x)}{\phi_1(x)} - a_1}{\phi_2(x)/\phi_1(x)} = \lim_{x \rightarrow x_0} \frac{(f(x)/\phi_1(x))'}{(\phi_2(x)/\phi_1(x))'},$$

as the last quotient is ultimately monotonic.

(iii). Inequalities (6.12), (6.13), (6.14) follow from the types of monotonicity of the involved functions, whereas (6.15) follows from, say, representation (4.24) with $m = a_2$, and from (6.14). The last assertion about $R(x)$ again follows from (4.24), or equivalently from (4.25), and the following two facts: the function $R(x)/\phi_1(x)$ tends to zero, as $x \rightarrow x_0$, and is either constant or strictly monotonic on some left neighborhood of x_0 . \square

8. Example: the special case of powers.

The case discussed in this section also serves as an illustration for the theory developed in [6; 7]. To fix the ideas we suppose $f \in AC^1]0, +\infty)$ and study the validity of asymptotic expansions of type

$$(8.1) \quad f(x) = a_1x^{\alpha_1} + a_2x^{\alpha_2} + o(x^{\alpha_2}), \quad x \rightarrow +\infty$$

$$(8.2) \quad f(x) = a_2x^{\alpha_2} + a_1x^{\alpha_1} + o(x^{\alpha_1}), \quad x \rightarrow 0^+ \quad (\alpha_1 > \alpha_2),$$

where α_1, α_2 are arbitrary real numbers. The associated (Euler) differential operator is

$$(8.3) \quad L_{\alpha_1, \alpha_2}[u] := u'' + (1 - \alpha_1 - \alpha_2)x^{-1}u' + \alpha_1\alpha_2x^{-2}u'', \quad x > 0.$$

As convenient canonical factorizations we may use the following

$$(8.4) \quad L_{\alpha_1, \alpha_2}[u] \equiv x^{\alpha_1-1} \left[x^{\alpha_2-\alpha_1+1} (x^{-\alpha_2}u)' \right]' \begin{cases} \nearrow \text{type (I) at } +\infty \\ \searrow \text{type (II) at } 0^+ \end{cases};$$

$$(8.5) \quad L_{\alpha_1, \alpha_2}[u] \equiv x^{\alpha_2-1} \left[x^{\alpha_1-\alpha_2+1} (x^{-\alpha_1}u)' \right]' \begin{cases} \nearrow \text{type (II) at } +\infty \\ \searrow \text{type (I) at } 0^+ \end{cases}.$$

The simple choice $T = T_0 = 1$ in the formulas of §4 is an admissible and convenient one for all values of α_i

Case : $x \rightarrow +\infty$. The basic quantities of our theory are:

$$(8.6) \quad \begin{cases} \phi_1(x) := x^{\alpha_1} ; \phi_2(x) := x^{\alpha_2} ; W(x) = (\alpha_2 - \alpha_1)x^{\alpha_1+\alpha_2-1} \quad (x > 0); \\ \Phi(x) = x^{\alpha_1} - x^{\alpha_2}. \end{cases}$$

$$(8.7) \quad f_1^*(x) = \frac{-x^{\alpha_2}f'(x) + \alpha_2x^{\alpha_2-1}f(x)}{(\alpha_2 - \alpha_1)x^{\alpha_1+\alpha_2-1}} = \frac{-xf'(x) + \alpha_2f(x)}{(\alpha_2 - \alpha_1)x^{\alpha_1}} =$$

$$\stackrel{\text{by (5.4)}}{=} \bar{c}_1 + \frac{1}{\alpha_1 - \alpha_2} \int_1^x t^{1-\alpha_1} \cdot L_{\alpha_1, \alpha_2}[f(t)]dt, \quad \bar{c}_1 = \alpha_2f(1) - f'(1);$$

$$(8.8) \quad f_2^*(x) = \frac{x^{\alpha_1}f'(x) - \alpha_1x^{\alpha_1-1}f(x)}{(\alpha_2 - \alpha_1)x^{\alpha_1+\alpha_2-1}} = \frac{xf'(x) + \alpha_1f(x)}{(\alpha_2 - \alpha_1)x^{\alpha_2}} =$$

$$\stackrel{\text{by (5.5)}}{=} \bar{c}_2 + \frac{1}{\alpha_2 - \alpha_1} \int_1^x t^{1-\alpha_2} \cdot L_{\alpha_1, \alpha_2}[f(t)]dt, \quad \bar{c}_2 = f'(1) - \alpha_1f(1);$$

$$(8.9) \quad F^*(x) = \frac{W(\Phi(x), f(x))}{W(x)} = \frac{(x^{\alpha_1} - x^{\alpha_2})f'(x) - (\alpha_1x^{\alpha_1-1} - \alpha_2x^{\alpha_2-1})f(x)}{(\alpha_2 - \alpha_1)x^{\alpha_1+\alpha_2-1}} =$$

$$\stackrel{\text{by (5.6)}}{=} f(1) + \frac{1}{\alpha_2 - \alpha_1} \int_1^x \frac{t^{\alpha_1} - t^{\alpha_2}}{t^{\alpha_1+\alpha_2-1}} \cdot L_{\alpha_1, \alpha_2}[f(t)]dt.$$

Specializing our theory we get the following results.

Proposition 8.1. Part I. *The following are equivalent properties:*

(i) *The pair of asymptotic expansions*

$$(8.10) \quad \begin{cases} f(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + o(x^{\alpha_2}) \\ f'(x) = a_1 \alpha_1 x^{\alpha_1-1} + a_2 \alpha_2 x^{\alpha_2-1} + o(x^{\alpha_2-1}) \end{cases}, x \rightarrow +\infty;$$

(ii) *The pair of asymptotic expansions*

$$(8.11) \quad \begin{cases} f(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + o(x^{\alpha_2}) \\ (x^{-\alpha_1} f(x))' = a_2 (\alpha_2 - \alpha_1) x^{\alpha_2 - \alpha_1 - 1} + o(x^{\alpha_2 - \alpha_1 - 1}) \end{cases}, x \rightarrow +\infty;$$

(iii) *The pair of asymptotic expansions*

$$(8.12) \quad \begin{cases} f(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + o(x^{\alpha_2}) \\ (x^{-\alpha_2} f(x))' = a_1 (\alpha_1 - \alpha_2) x^{\alpha_1 - \alpha_2 - 1} + o(x^{-1}) \end{cases}, x \rightarrow +\infty;$$

(iv) *The improper integral*

$$(8.13) \quad \int_1^{+\infty} t^{1-\alpha_2} \cdot L_{\alpha_1, \alpha_2}[f(t)] dt \equiv \int_1^{+\infty} [t^{\alpha_1 - \alpha_2 + 1} (t^{-\alpha_1} f(t))']' dt \text{ converges.}$$

And to this list we may add the geometric properties in theorem 4.5 concerning the limits of f_1^*, f_2^*, F^* .

Part II. Whenever properties in part I hold true then

$$(8.14) \quad a_1 = \alpha_2 f(1) - f'(1) + \frac{1}{\alpha_1 - \alpha_2} \int_1^{+\infty} [t^{\alpha_2 - \alpha_1 + 1} (t^{-\alpha_2} f(t))']' dt;$$

$$(8.15) \quad a_2 = -\alpha_1 f(1) + f'(1) + \frac{1}{\alpha_2 - \alpha_1} \int_1^{+\infty} [t^{\alpha_1 - \alpha_2 + 1} (t^{-\alpha_1} f(t))']' dt;$$

and we have the following representations

$$(8.16) \quad f(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + x^{\alpha_1} \cdot \int_x^{+\infty} t^{\alpha_2 - \alpha_1 - 1} dt \int_t^{+\infty} s^{1-\alpha_2} \cdot L_{\alpha_1, \alpha_2}[f(s)] ds;$$

$$(8.17) \quad f(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + x^{\alpha_2} \cdot \int_x^{+\infty} t^{\alpha_1 - \alpha_2 - 1} dt \int_t^{+\infty} s^{1-\alpha_1} \cdot L_{\alpha_1, \alpha_2}[f(s)] ds.$$

Part III. In the special case wherein the quantity $L_{\alpha_1, \alpha_2}[f(x)]$ is one-signed (≥ 0 or ≤ 0) for all x large enough then to all the equivalent properties listed in part I the following can be added:

$$(8.18) \quad f(x) = a_1 x^{\alpha_1} + O(x^{\alpha_2}) \quad , \quad x \rightarrow +\infty ;$$

$$(8.19) \quad f(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + o(x^{\alpha_2}) \quad , \quad x \rightarrow +\infty .$$

Here the import is that the sole relation (8.18) automatically implies the pair (8.10).

Part IV. The following are equivalent properties

(v) The pair of asymptotic expansions

$$(8.20) \quad \begin{cases} f(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + o(x^{\alpha_2}) \\ f'(x) = a_1 \alpha_1 x^{\alpha_1-1} + o(x^{\alpha_1-1}) \end{cases} \quad , \quad x \rightarrow +\infty ;$$

(vi) The pair of asymptotic expansions

$$(8.21) \quad \begin{cases} f(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + o(x^{\alpha_2}) \\ (x^{-\alpha_2} f(x))' = a_1(\alpha_1 - \alpha_2) x^{\alpha_1 - \alpha_2 - 1} + o(x^{\alpha_1 - \alpha_2 - 1}) \end{cases} \quad , \quad x \rightarrow +\infty ;$$

(vii) The improper integral

$$(8.22) \quad \int_1^{+\infty} t^{\alpha_1 - \alpha_2 - 1} dt \int_t^{+\infty} s^{1 - \alpha_1} \cdot L_{\alpha_1, \alpha_2}[f(s)] ds \text{ converges} .$$

The two equivalences “(8.10) \Leftrightarrow (8.11)” and “(8.20) \Leftrightarrow (8.21)” are not contained in the theory developed in this paper but are simple algebraic facts that can be directly checked for any numbers $\alpha_1, \alpha_2 \in \mathbb{R}$. The corresponding proofs for n -term expansions in real powers are to be found in [7; lemmas 7.3 and 7.4]. In this case the factorizational approach gives characterizations of standard differentiation of an asymptotic expansion, i.e. differentiation obtained by the application of the operator d/dx .

This fact can be extended to a larger class of asymptotic expansions using the concept of regular variation but it will not be investigated here.

For the elementary case of asymptotic straight lines, i.e. $\alpha_1 = 1$ and $\alpha_2 = 0$, we have the characterizations

$$(8.23) \quad \begin{cases} f(x) = a_1 x + a_2 + o(1) \\ f'(x) = a_1 + o(1) \end{cases} \quad , x \rightarrow +\infty \quad \Leftrightarrow \quad \int_1^{+\infty} dt \int_t^{+\infty} f''(s) ds \text{ converges} ;$$

$$(8.24) \quad \begin{cases} f(x) = a_1 x + a_2 + o(1) \\ f'(x) = a_1 + o(x^{-1}) \end{cases} \quad , x \rightarrow +\infty \quad \Leftrightarrow \quad \int_1^{+\infty} t f''(t) dt \text{ converges} .$$

The pair of asymptotic relations in (8.23) may be labelled by the locution “*the graph of f admits of the straight line $y = a_1x + a_2$ as a first-order asymptote at $+\infty$* ”. The pair in (8.24) states the fact that the straight line $y = a_1x + a_2$ is “*the limit tangent (\equiv asymptotic tangent) at $+\infty$* ”.

Case: $x \rightarrow 0^+$. The basic quantities are the same as in the foregoing case with the roles of α_1, α_2 interchanged; and each integral of type, say $\int_1^{+\infty} \dots$ must be replaced by the integral $\int_{\rightarrow 0}^1 \dots$ of the same quantity, wherein the endpoint “0” is the sole possible singularity. We leave to the reader the complete formulation of the corresponding version of proposition 8.1 mentioning only the two main equivalences.

The pair of expansions

$$(8.25) \quad \begin{cases} f(x) = a_2x^{\alpha_2} + a_1x^{\alpha_1} + o(x^{\alpha_1}) \\ f'(x) = a_2\alpha_2x^{\alpha_2-1} + a_1\alpha_1x^{\alpha_1-1} + o(x^{\alpha_1-1}) \end{cases}, \quad x \rightarrow 0^+, (\alpha_2 < \alpha_1),$$

holds true iff

$$(8.26) \quad \int_{\rightarrow 0}^1 t^{1-\alpha_1} \cdot L_{\alpha_1, \alpha_2}[f(t)]dt \equiv \int_{\rightarrow 0}^1 [t^{\alpha_2-\alpha_1+1}(t^{-\alpha_2}f(t))']'dt \text{ converges};$$

whereas the pair of expansions

$$(8.27) \quad \begin{cases} f(x) = a_2x^{\alpha_2} + a_1x^{\alpha_1} + o(x^{\alpha_1}) \\ f'(x) = a_2\alpha_2x^{\alpha_2-1} + o(x^{\alpha_2-1}) \end{cases}, \quad x \rightarrow 0^+, (\alpha_2 < \alpha_1),$$

holds true iff

$$(8.28) \quad \int_{\rightarrow 0}^1 t^{\alpha_2-\alpha_1-1}dt \int_0^t s^{1-\alpha_2} \cdot L_{\alpha_1, \alpha_2}[f(s)]ds \text{ converges}.$$

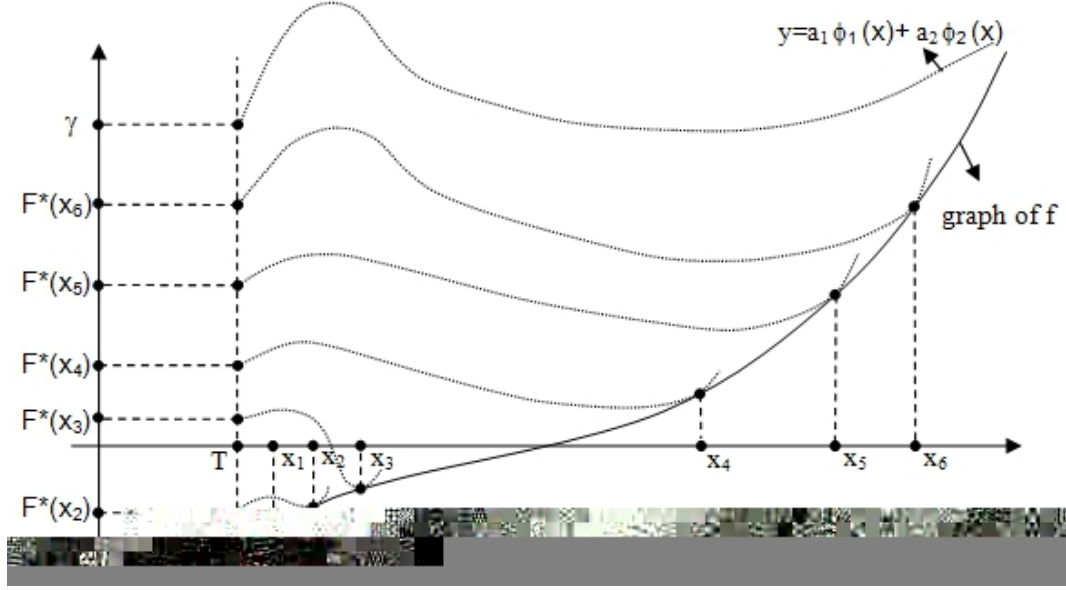
For $\alpha_2 = 0$ and $\alpha_1 = 1$ (8.25) reduces to

$$(8.29) \quad \begin{cases} f(x) = a_2 + a_1x + o(x) \\ f'(x) = a_1 + o(1) \end{cases}, \quad x \rightarrow 0^+,$$

which is obviously equivalent to

$$(8.30) \quad f'(x) = a_1 + o(1), \quad x \rightarrow 0^+,$$

that is to say, to the existence of a finite limit: $\lim_{x \rightarrow 0^+} f'(x)$. Condition (8.26) reduces to the convergence of $\int_{\rightarrow 0}^1 f''(t)dt$, and this condition is equivalent to (8.30) under our present assumption $f \in AC^1]0, 1]$.



This is just the simple technical idea underlying our factorizational theory together with the theory of canonical factorizations which yield the means for applying the simple idea to general expansions.

In closing this paper we present a figure illustrating the concept of limit tangent curve, as characterized in theorem 4.5, for a generalized convex function.

The figure refers to the following contingency:

1. $\phi_1, \phi_2, f \in C^2[T, +\infty)$; $\phi_2 > 0$; $f \in \mathcal{C}(\phi_1, \phi_2; [T, +\infty))$, hence F^* is increasing by theorem 6.1-(ii).
2. Each dotted curve, save the uppermost, has a first-order contact with the graph of f at a point x_i and has equation

$$y = f_1^*(x_i)\phi_1(x) + f_2^*(x_i)\phi_2(x).$$

3. $F^*(x_i)$ is the contact indicatrix of order one at the point x_i with respect to the family $\mathcal{F} := \text{span}(\phi_1, \phi_2)$ and to the line $x = T$.
4. The $\lim_{x \rightarrow +\infty} F^*(x) = \gamma$ exists in \mathbb{R} .

From these facts the following follow:

- (i) The two limits

$$\lim_{x \rightarrow +\infty} f_1^*(x) \equiv a_1, \quad \lim_{x \rightarrow +\infty} f_2^*(x) \equiv a_2,$$

exist in \mathbb{R} and are linked to γ by relation $\gamma = a_1\phi_1(T) + a_2\phi_2(T)$.

- (ii) The uppermost dotted curve, whose equation is $y = a_1\phi_1(x) + a_2\phi_2(x)$, is by definition the limit tangent curve to the graph of f with respect to the family \mathcal{F} as $x \rightarrow +\infty$.

(iii) The asymptotic relations hold true:

$$\begin{cases} f(x) = a_1\phi_1(x) + a_2\phi_2(x) + o(\phi_2(x)), x \rightarrow +\infty, \\ \left(\frac{f(x)}{\phi_1(x)}\right)' = a_2 \left(\frac{\phi_2(x)}{\phi_1(x)}\right)' + o\left[\left(\frac{\phi_2(x)}{\phi_1(x)}\right)'\right], x \rightarrow +\infty. \end{cases}$$

9. Formal differentiation of a two-term asymptotic expansion: a Tauberian result of interpolatory type

Here we examine the classical problem of looking for conditions under which an expansion

$$(9.1) \quad f(x) = a_1\phi_1(x) + a_2\phi_2(x) + o(\phi_2(x)), x \rightarrow x_0,$$

implies

$$(9.2) \quad f'(x) = a_1\phi_1'(x) + a_2\phi_2'(x) + o(\phi_2'(x)), x \rightarrow x_0.$$

The reader is referred to the introduction in [6] highlighting the inherent differences between the problems of differentiating in some formal sense an asymptotic relation $f(x) = \phi(x) + o(\phi(x))$ or an asymptotic expansion with two meaningful terms.

For our problem the first remark is that theorems in §§4,5 show that the differentiated relation (9.2) is in general no good match for (9.1). As a simple example take the function

$$(9.3) \quad g_1(x) := e^x + x + \sin x = e^x + x + o(x), x \rightarrow +\infty,$$

defined, say, on $[2, +\infty)$ for which we have:

$$(9.4) \quad g_1^{(k)}(x) = e^x + O(1), x \rightarrow +\infty; k \geq 1.$$

Hence the asymptotic expansion (9.3) is not formally differentiable if standard derivatives are used; however we have:

$$(9.5) \quad g_1(x)/e^x = 1 + xe^{-x} + \sin x \cdot e^{-x} = 1 + xe^{-x}[1 + o(1)], x \rightarrow +\infty,$$

$$(9.6) \quad (g_1(x)/e^x)' = (1-x)e^{-x} + (\cos x - \sin x)e^{-x} = (1-x)e^{-x}[1 + o(1)], x \rightarrow +\infty.$$

Putting $\phi_1(x) := e^x$, $\phi_2(x) := x$ relation (9.6) is just (4.29)₂. The operator L associated to the pair (e^x, x) is

$$(9.7) \quad L[u] := u'' + \frac{x}{1-x}u' - \frac{1}{1-x}u \text{ on }]1, +\infty),$$

and the integral in (5.15) becomes

$$(9.8) \quad \int_{-1}^{+\infty} \frac{e^x}{(1-x)e^x} L[g(x)] dx = \int_{-1}^{+\infty} \left[\frac{-2 \sin x + x(\sin x + \cos x)}{(1-x)^2} \right] dx,$$

which is convergent by Abel's test, according to Theorem 5.2-(II). A second simple example, left to the reader, is that of

$$(9.9) \quad g_2(x) := x + e^{-x} + x^{-1} \sin x \cdot e^{-x} = x + e^{-x}[1 + o(1)], \quad x \rightarrow +\infty,$$

which is not formally differentiable; here g_2 has the properties in Theorem 4.3 but not those in Theorem 4.5.

Hence our theory puts in evidence the fact that the spontaneous choice of the operator d/dx not always is the right choice when formally differentiating an asymptotic expansion with at least two meaningful terms. In the framework of our theory it is one of the two operators $L_1[u] := (u/\phi_1)'$, $L_2[u] := (u/\phi_2)'$ which works well and the results in §5 characterize the pair (4.19) and the pair (4.29)_{1,2}. These results, being characterizations, completely settle the problem but the classical formulation of the interpolatory approach was a bit different as will be clearly shown by the case of real powers. By Proposition 8.1 the pair of relations

$$(9.10) \quad f(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + o(x^{\alpha_2}), \quad x \rightarrow +\infty, \quad (\alpha_1 > \alpha_2),$$

$$(9.11) \quad f'(x) = a_1 \alpha_1 x^{\alpha_1-1} + a_2 \alpha_2 x^{\alpha_2-1} + o(x^{\alpha_2-1}), \quad x \rightarrow +\infty, \quad (\alpha_1 > \alpha_2),$$

is characterized by the integral condition in (8.13) with L_{α_1, α_2} defined in (8.3). Hence knowing (9.10) to be true, relation (9.11) holds true iff (8.13) is satisfied. This is certainly the case if

$$(9.12) \quad L_{\alpha_1, \alpha_2} = O(x^{\alpha_2-2-\epsilon}), \quad x \rightarrow +\infty, \quad \text{for some } \epsilon > 0;$$

but in the study of the n -body problem, for instance, it is of interest to grant (9.11) under the weaker condition

$$(9.13) \quad L_{\alpha_1, \alpha_2} = O(x^{\alpha_2-2}), \quad x \rightarrow +\infty,$$

and this is no elementary question.

The problem can be posed in a technically different way by putting

$$R(x) := f(x) - a_1 x^{\alpha_1} - a_2 x^{\alpha_2}$$

and inferring $R'(x) = o(x^{\alpha_2-1})$ from both relations

$$R(x) = o(x^{\alpha_2}), \quad R''(x) = O(x^{\alpha_2-2}), \quad x \rightarrow +\infty.$$

This inference is known to be true: an “ $\epsilon - \delta$ ”-proof may be found, e.g., in Boas [15] for a twice-differentiable function and in Saari [16] for an $f \in AC^1[T, +\infty)$.

Now condition $R''(x) = O(x^{\alpha_2-2})$ is algebraically natural whereas condition (9.13) naturally follows from our factorizational theory. But in the case of a general asymptotic expansion there is no algebraic evidence and it is our theory that leads to formulate the appropriate

Conjecture on the formal differentiation of a general two-term expansion from the classical interpolatory standpoint. *Let a function $f \in AC^1[T, x_0[$ admit of*

an asymptotic expansion (9.1) under our strenghtened basic asssumptions (3.1) and (3.4). It follows from Theorems 4.5 and 5.2-(II) that (9.1) is formally differentiable in the sense of (4.29)₂ iff

$$(9.14) \quad \int_T^{x_0} \phi_1(t)(W(t))^{-1} \cdot L[f(t)]dt \quad \text{converges}.$$

We now suggest the following heuristic considerations. It is implicit in the use of an asymptotic expansion (9.1) that we are measuring our quantities by means of the given functions ϕ_1, ϕ_2 and of their ratios for which we know that $\phi_2/\phi_1 = o(1)$ and that $\int^{x_0} (\phi_2/\phi_1)'$ converges. Hence in this context, if the integrand in (9.14) is $O((\phi_2(x)/\phi_1(x))')$ i.e.

$$(9.15) \quad L[f(x)] = O\left([\phi_2(x)/\phi_1(x)]^2 \cdot \phi_1(x)\right) \equiv O((W(x))^2(\phi_1(x))^{-3}), \quad x \rightarrow x_0,$$

then (9.14) is satisfied; but if this integrand satisfies

$$(9.16) \quad \phi_1(x)(W(x))^{-1} \cdot L[f(x)] = \left\{ \begin{matrix} O \\ o \end{matrix} \right\} \left(\frac{\phi_1(x)}{\phi_2(x)} \left(\frac{\phi_2(x)}{\phi_1(x)} \right)' \right), \quad x \rightarrow x_0,$$

that is to say

$$(9.17) \quad L[f(x)] = O\left([\phi_2(x)/\phi_1(x)]^2(\phi_1(x)^2(\phi_2(x))^{-1})\right) \equiv O((W(x))^2(\phi_1(x))^{-2}(\phi_2(x))^{-1}), \quad x \rightarrow x_0,$$

then (9.14) is not automatically granted. This is precisely the Tauberian condition we wish to investigate conjecturing that it is sufficient for the inference “(9.1) \implies (4.29)₂”. In the case of powers $\phi_i \equiv x^{\alpha_i}$, $\alpha_1 > \alpha_2$, and as $x \rightarrow +\infty$, we have

$$(9.18) \quad \begin{cases} (\phi_2(x)/\phi_1(x))' = (\alpha_2 - \alpha_1)x^{\alpha_2 - \alpha_1 - 1}, \\ (\phi_1(x)/\phi_2(x)) \cdot (\phi_2(x)/\phi_1(x))' = (\alpha_2 - \alpha_1)x^{-1}. \end{cases}$$

We present here one of the possible results to show the usefulness of canonical factorizations in this context; the proof does not follow classical patterns but is based on an interplay between the two types of factorizations.

Theorem 9.1. *Hypotheses: (i) the strenghtened basic asssumptions (3.1) and (3.4); (ii) $f \in AC^1[T, x_0[$ and the expansion (9.1) i.e. (4.29)₁; (iii) L is the operator defined by (3.5)-(3.6). Thesis:*

- (I) If (9.17) is satisfied with “ O ” replaced by “ o ” then (4.29)₂ holds true.
- (II) If (9.17) is satisfied then, in general, only the weaker relation holds true:

$$(9.19) \quad (f(x)/\phi_1(x))' = O((\phi_2(x)/\phi_1(x))'), \quad x \rightarrow x_0.$$

But if the ratio $\phi := \phi_2/\phi_1$ satisfies the additional conditions that ϕ' is strictly one-signed on a neighborhood of x_0 and

$$(9.20) \quad \begin{cases} (x - x_0)\phi'(x) \asymp \phi(x) \\ (x - x_0)\phi''(x) \asymp \phi'(x) \end{cases}, \quad x \rightarrow x_0 \quad (\text{if } x_0 \in \mathbb{R}),$$

or

$$(9.21) \quad \begin{cases} x\phi'(x) \asymp \phi(x) \\ x\phi''(x) \asymp \phi'(x) \end{cases}, \quad x \rightarrow +\infty \quad (\text{if } x_0 = +\infty),$$

then (4.29)₂ holds true.

Notation $g_1(x) \asymp g_2(x)$, $x \rightarrow x_0$, denotes the validity of both relations

$$(9.22) \quad g_1(x) = O(g_2(x)); \quad g_2(x) = O(g_1(x)), \quad x \rightarrow x_0.$$

Conditions (9.20) and (9.21) respectively imply

$$(9.23) \quad (\phi'(x))^2 \asymp \phi(x)\phi''(x), \quad \begin{cases} x \rightarrow x_0 \\ x \rightarrow +\infty \end{cases},$$

which will be essential in the proof.

Proof. (I) Put

$$(9.24) \quad M_1(x) := \phi_1(x)(W(x))^{-1} \cdot L[f(x)]; \quad M_2(x) := \phi_2(x)(W(x))^{-1} \cdot L[f(x)].$$

Condition (9.17) with “ O ” replaced by “ o ” is equivalent to each one of the following:

$$(9.25) \quad M_1(x) := o\left(\frac{\phi_1(x)}{\phi_2(x)} \left(\frac{\phi_2(x)}{\phi_1(x)}\right)'\right) \equiv o\left(\frac{W(x)}{\phi_1(x)\phi_2(x)}\right), \quad x \rightarrow x_0;$$

$$(9.26) \quad M_2(x) := o\left(\left(\frac{\phi_2(x)}{\phi_1(x)}\right)'\right) \equiv o\left(\frac{W(x)}{(\phi_1(x))^2}\right), \quad x \rightarrow x_0.$$

Condition (9.25) by itself does grant neither (5.15) nor (4.14) but (9.26) grants (5.12) and so we may rewrite representation (5.9) in the form:

$$(9.27) \quad \begin{cases} f(x) = c_1\phi_1(x) + c_2\phi_2(x) + \phi_2(x) \int_T^x (\phi_1(t)/\phi_2(t))' dt \int_t^{x_0} M_2(s) ds = \\ = c_1\phi_1(x) + c_2\phi_2(x) + \phi_2(x) \cdot o\left(\frac{\phi_1(x)}{\phi_2(x)}\right) = c_1\phi_1(x) + o(\phi_1(x)), \quad x \rightarrow x_0. \end{cases}$$

From (9.27) and (9.1) we get both $c_1 = a_1$ and

$$(9.28) \quad \begin{cases} \int_T^x (\phi_1(t)/\phi_2(t))' dt \int_t^{x_0} M_2(s) ds = [f(x) - a_1\phi_1(x) - c_2\phi_2(x)]/\phi_2(x) = \\ = [(a_2 - c_2)\phi_2(x) + o(\phi_2(x))]/\phi_2(x) = (a_2 - c_2) + o(1), \quad x \rightarrow x_0. \end{cases}$$

Hence (5.14) holds true and, instead of (9.27) we may use representation

$$(9.29) \quad f(x) = a_1\phi_1(x) + a_2\phi_2(x) - \phi_2(x) \int_x^{x_0} (\phi_1(t)/\phi_2(t))' dt \int_t^{x_0} M_2(s) ds, \quad x \in [T, x_0[,$$

where a_1, a_2 are the same coefficients as in (4.29)₁. Moreover Theorem 5.2-(I) implies the formally-differentiated expansion appearing in (4.19) but we want to prove the stronger relation (4.29)₂. From (9.29) we get

$$(9.30) \quad \begin{cases} (f/\phi_1)' = a_2(\phi_2/\phi_1)' - (\phi_2/\phi_1)' \cdot \int_x^{x_0} (\phi_1/\phi_2)' dt \int_t^{x_0} M_2(s) ds + \\ + (\phi_2/\phi_1) \cdot (\phi_2/\phi_1)' \cdot \int_x^{x_0} M_2(s) ds, \end{cases}$$

where, by (9.26), the last term in the right-hand side satisfies the estimate:

$$(9.31) \quad \begin{cases} \frac{\phi_2(x)}{\phi_1(x)} \left(\frac{\phi_1(x)}{\phi_2(x)} \right)' \cdot \int_x^{x_0} M_2(s) ds \equiv - \frac{\phi_1(x)}{\phi_2(x)} \left(\frac{\phi_2(x)}{\phi_1(x)} \right)' \cdot \int_x^{x_0} M_2(s) ds = \\ = \frac{\phi_1(x)}{\phi_2(x)} \left(\frac{\phi_2(x)}{\phi_1(x)} \right)' \cdot o \left(\frac{\phi_2(x)}{\phi_1(x)} \right) = o \left(\left(\frac{\phi_2(x)}{\phi_1(x)} \right)' \right), \quad x \rightarrow x_0. \end{cases}$$

Relation (4.29)₂ follows at once from (9.30), (9.31). (II) The above calculations are valid until (9.30), and in (9.31) “ o ” is replaced by “ O ” so that we can only infer (9.19). Now, under the additional conditions, we put

$$(9.32) \quad h(x) := \frac{f(x)}{\phi_1(x)} - a_1 - a_2\phi(x), \quad \text{where } \phi(x) := \phi_2(x)/\phi_1(x),$$

so that (9.1) is equivalently written as

$$(9.33) \quad h(x) = o(\phi(x)), \quad x \rightarrow x_0,$$

and (9.30) yields

$$(9.34) \quad h'(x) \equiv \left(\frac{f(x)}{\phi_1(x)} \right)' - a_2\phi'(x) = O(\phi'(x)), \quad x \rightarrow x_0.$$

To prove $h'(x) = o(\phi'(x))$ we try to use some interpolatory-type result on formal differentiation by evaluating h'' by a suitable device. Replacing $\phi' = W \cdot (\phi_1)^{-2}$ in factorization (5.10) we get

$$(9.35) \quad L[f] = \frac{W}{\phi_1} \left[\frac{1}{\phi'} \left(\frac{f}{\phi_1} \right)' \right]' = \frac{W}{\phi_1} \left[\left(\frac{1}{\phi'} \right)' \left(\frac{f}{\phi_1} \right)' + \frac{1}{\phi'} \left(\frac{f}{\phi_1} \right)'' \right],$$

whence

$$(9.36) \quad \begin{cases} \left(\frac{f}{\phi_1} \right)'' = \phi' \left[\frac{\phi_1}{W} L[f] - \left(\frac{1}{\phi'} \right)' \left(\frac{f}{\phi_1} \right)' \right] = \frac{\phi_1\phi'}{W} L[f] + \frac{\phi''}{\phi'} \left(\frac{f}{\phi_1} \right)' = \\ \text{by (9.17) and (9.19)} \quad O \left(\frac{\phi'W}{\phi_1\phi_2} \right) + O(\phi'') = O \left(\frac{(\phi')^2}{\phi} \right) + O(\phi'') \quad \text{by (9.23)} \quad O(\phi''), \end{cases}$$

and

$$(9.37) \quad h'' \equiv \left(\frac{f}{\phi_1} \right)'' - a_2 \phi'' = O(\phi'').$$

Hence h satisfies “ $h = o(\phi)$, $h'' = O(\phi'')$ ”. The assumptions on ϕ make applicable a result by Boas [15; th. 1B, p. 638] so inferring

$$(9.38) \quad h'(x) = o(|\phi(x)\phi''(x)|^{1/2}) \stackrel{\text{by (9.23)}}{=} o(\phi'(x)), \quad x \rightarrow x_0.$$

□

The original proof given by Boas is for a function h of class C^2 but it can be easily adapted to the case $h \in AC^1$.

As far as (9.21) is concerned we point out that the meaning of a condition like

$$(9.39) \quad x\phi'(x) \asymp \phi(x), \quad x \rightarrow +\infty, \quad x \rightarrow x_0,$$

is properly understood in the context of regular variation. Referring to the monograph by Bingham, Goldie and Teugels [14; th. 2.2.6, p. 74] the positive absolutely-continuous functions ϕ satisfying (9.39) are a subclass of the so-called “*extended regularly-varying functions at $+\infty$* ” in the sense of Karamata. A still narrower class is obviously that of functions ϕ such that

$$(9.40) \quad \phi'(x)/\phi(x) = \alpha x^{-1} + o(x^{-1}), \quad x \rightarrow +\infty, \quad \text{for some } \alpha \in \mathbb{R},$$

which may be called “*regularly-varying functions at $+\infty$ in a strong sense with index α* ”.

The analogous class at a point x_0^- , where $x_0 \in \mathbb{R}$, is defined requiring that the associated functions $\bar{\phi}(x) := \phi((x_0 - x)^{-1})$ satisfies (9.40). For such classes of functions a nice result states the equivalence between the pair (4.29)_{1,2} and the pair (9.1)-(9.2). We need an intermediary result.

Proposition 9.2. *Let hypotheses (4.13) hold true; under the additional assumptions:*

$$(9.41) \quad \phi'_2(x) = o(\phi'_1(x)), \quad x \rightarrow x_0,$$

$$(9.42) \quad \phi'_1(x)/\phi_1(x) = O(\phi'_2(x)/\phi_2(x)), \quad x \rightarrow x_0,$$

the pair (4.29)_{1,2} implies the pair (9.1)-(9.2).

Proof. From (4.29)₂ we get:

$$f'\phi_1 = f\phi'_1 + a_2[\phi'_2\phi_1 - \phi_2\phi'_1] + o(\phi'_2\phi_1 - \phi_2\phi'_1) \stackrel{(4.29)_1}{=} a_1\phi_1\phi'_1 + a_2\phi'_2\phi_1 + o(\phi'_2\phi_1) + o(\phi_2\phi'_1),$$

from whence

$$(9.43) \quad f' = a_1\phi'_1 + a_2\phi'_2 + o(\phi'_2) + o(\phi_2\phi'_1/\phi_1) \stackrel{(9.42)}{=} a_1\phi'_1 + a_2\phi'_2 + o(\phi'_2).$$

□

Theorem 9.3. *Let hypotheses (4.13) and (9.41) hold true together with the following additional assumption:*

$$(9.44) \quad \begin{cases} \phi_1, \phi_2 \text{ regularly - varying at } x_0^- \text{ in a strong sense} \\ \text{with respective indexes } \alpha_1, \alpha_2; \alpha_1 > \alpha_2, \alpha_2 \neq 0. \end{cases}$$

Then the pair (4.29)_{1,2} is equivalent to the pair (9.1)-(9.2).

Proof. Case: $x_0 = +\infty$. From (9.40) applied to ϕ_1, ϕ_2 we easily infer the following relations:

$$(9.45) \quad (\phi_2(x)/\phi_1(x))' \sim (\alpha_2 - \alpha_1)x^{-1}\phi_2(x)/\phi_1(x), \quad x \rightarrow +\infty;$$

$$(9.46) \quad \phi_1'(x)/\phi_1(x) \sim (\alpha_1/\alpha_2)\phi_2'(x)/\phi_2(x), \quad x \rightarrow +\infty, \quad \text{if } \alpha_1, \alpha_2 \neq 0;$$

$$(9.47) \quad \phi_1'(x)/\phi_1(x) = o(\phi_2'(x)/\phi_2(x)), \quad x \rightarrow +\infty, \quad \text{if } \alpha_1 = 0, \alpha_2 \neq 0.$$

Now the inference “(4.29)_{1,2} \implies (9.1)-(9.2)” follows from Proposition 9.2 as condition (9.42) is implied by (9.46)-(9.47). Viceversa from (9.1)-(9.2) we get:

$$(9.48) \quad f'\phi_1 - f\phi_1' = a_2[\phi_2'\phi_1 - \phi_2\phi_1'] + o(\phi_1\phi_2') + o(\phi_1'\phi_2),$$

from whence

$$(9.49) \quad (f/\phi_1)' = a_2(\phi_2/\phi_1)' + o(\phi_2'/\phi_1) + o(\phi_1'\phi_2/(\phi_1)^2).$$

In the case $\alpha_1, \alpha_2 \neq 0$ we get:

$$(9.50) \quad \phi_2'/\phi_1 \stackrel{(9.46)}{\sim} \frac{\alpha_2}{\alpha_1}\phi_1'\phi_2/(\phi_1)^2 \stackrel{(9.44)}{\sim} \alpha_2x^{-1}\phi_2/\phi_1 \stackrel{(9.45)}{\sim} \frac{\alpha_2}{\alpha_2 - \alpha_1}(\phi_2/\phi_1)',$$

and (4.29)₂ follows from (9.49) and (9.450). In the case $\alpha_1 = 0$ we have:

$$(9.51) \quad \begin{cases} \phi_1'\phi_2/(\phi_1)^2 \stackrel{(9.47)}{=} o(\phi_2'/\phi_1), \\ \phi_2'/\phi_1 \stackrel{(9.44)}{\sim} \alpha_2x^{-1}\phi_2/\phi_1 \stackrel{(9.45)}{\sim} \frac{\alpha_2}{\alpha_2 - \alpha_1}(\phi_2/\phi_1)', \end{cases}$$

and (4.29)₂ follows from (9.49) and (9.51). The case $x_0 \in \mathbb{R}$ reduces to the case $x_0 = +\infty$ by the mentioned change of variable: asymptotic relations as $x \rightarrow x_0^-$ change into equivalent asymptotic relations as $x \rightarrow +\infty$, at least in our present situation involving only first-order derivatives. \square

Remark. The additional conditions in the last two propositions are merely sufficient for the respective theses. In the elementary case of the scale “ $x \gg 1, x \rightarrow +\infty$ ”, it happens that the pair (4.29)_{1,2} implies the pair (9.1)-(9.2), and even the stronger relation $f' = a_1 + o(x^{-1})$, though (9.42) does not hold.

References

- [1] W. A. COPPEL, *Disconjugacy*. Lecture Notes in Mathematics, vol. 220. Springer-Verlag, Berlin, 1971.
- [2] J. DIEUDONNE, *Calcul infinitésimal*. Hermann, Paris, 1968.
- [3] A. GRANATA, Canonical factorizations of disconjugate differential operators, *SIAM J. Math. Anal.*, **11** (1980), 160-172.
- [4] A. GRANATA, Canonical factorizations of disconjugate differential operators -Part II, *SIAM J. Math. Anal.*, **19** (1988), 1162-1173.
- [5] A. GRANATA, Polynomial asymptotic expansions in the real domain: the geometric, the factorizational, and the stabilization approaches, *Analysis Mathematica*, **33** (2007), 161-198.
- [6] A. GRANATA, The problem of differentiating an asymptotic expansion in real powers. Part I: Unsatisfactory or partial results by classical approaches, *Analysis Mathematica*, **36** (2010), 85-112.
- [7] A. GRANATA, The problem of differentiating an asymptotic expansion in real powers. Part II: factorizational theory, *Analysis Mathematica*, **36** (2010), 173-218.
- [8] O. HAUPT, Über Asymptoten ebener Kurven, *J. Reine Angew. Math.*, **152** (1922), 6-10; ibidem p.239.
- [9] E. HEWITT and K. STROMBERG, *Real and abstract Analysis*. Springer-Verlag, Berlin, Heidelberg, New York, 1969.
- [10] S. KARLIN and W. STUDDEN, *Tchebycheff systems: with applications in analysis and statistics*. Interscience, New York, 1966.
- [11] A. M. OSTROWSKI, Note on the Bernoulli-L'Hospital rule, *Amer. Math. Monthly*, **83** (1976), 239-242.
- [12] A. M. OSTROWSKI, On Cauchy-Frullani integrals, *Comm. Math. Helv.*, **51** (1976), 57-91.
- [13] W. F. TRENCH, Canonical forms and principal systems for general disconjugate equations, *Trans. Amer. Math. Soc.*, **189** (1974), 139-327.

Additional references for §9

- [14] N. H. BINGHAM, C. M. GOLDIE, and J. L. TEUGELS, *Regular variation*. Cambridge University Press, Cambridge, 1987.
- [15] R. P. BOAS Jr., Asymptotic relations for derivatives, *Duke Math. J.*, **3** (1937), 637-646.
- [16] D. G. SAARI, An elementary Tauberian theorem for absolutely continuous functions and for series, *SIAM J. Math. Anal.*, **(5)** (1974), 649-662.

Corrections of misprints in the published version of the present e-paper:

A. GRANATA, Analytic theory of finite asymptotic expansions in the real domain. Part I: two-term expansions of differentiable functions, *Analysis Mathematica*, **37**(2011), 245-287. DOI: 10.1007/s10476-011-0402-7.

On p. 251 in the unnumbered formula $f_i^*(t) = q_i(t) \cdot L[f(t)]$ correct the left-hand side into $(f_i^*)'(t)$.

On p. 255 in formula (4.4) the quantity $(f(t)/\phi_1(t_0))'$ must be read as $(f(t)/\phi_1(t))'$.

On p. 259 notice that relation (4.27) in Theorem 4.5 is a different formulation of relation $(4.29)_2$.

On p. 261, in the first line locution "*t limit position*" must be simply read "*limit position*".

On p. 263, hypothesis (ii) in Lemma 5.1 must be read " $f \in AC^1(I)$ " as stated at the outset of §5.

On p. 266, in each of the two formulas (5.27)-(5.28) there is a redundant 'absolute value' between the functions inside the integral.

On p. 284, inside the second integral in formula (8.26) the number " -1 " must be changed into " $+1$ ", hence the correct version of formula (8.26) is:

$$\int_{\rightarrow 0}^1 t^{1-\alpha_1} \cdot L_{\alpha_1, \alpha_2}[f(t)] dt \equiv \int_{\rightarrow 0}^1 [t^{\alpha_2-\alpha_1+1} (t^{-\alpha_2} f(t))']' dt \text{ converges.}$$

On p. 286: in reference [7] the page numbers of the paper are missing, namely "173-218".